TEMPERAMENT IN BACH'S *WELL-TEMPERED CLAVIER*

A historical survey and a new evaluation according to dissonance theory

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Foreword

My double training in engineering and music and also my special interest in tunings and temperaments since I was very young (perhaps as a consequence of my interest in maths and music) has led me to work in a mixed project in which I intend to give real solutions to musicology using mathematical tools and new technologies.

I started to develop certain mathematical theories for tuning and temperaments at the moment when I acquired suitable tools in my engineering studies in the first year of my degree, specifically algebraic tools such as vectorial spaces, discrete mathematical theories and others. Later, with the development of new mathematical theories for tunings and temperaments (like those by Joan Girbau,[1] Francisco Javier Sánchez[2] and Carlos García Suárez[3] among others in Spain and lots more abroad, like those by Easley Blackwood[4] and Mark Lindley,[5] for example), I obtained more tools to continue my research in this field. At the same time, I received a confirmation of the working of these complex mathematical theories. More mathematical theories about music have been developed in the last century, theories that are beyond my scope here, but one of them has an especial relevance for this work and it is the one developed by William A. Sethares.[6] This theory continues the studies and principles brought out by Hermann Ludwig Ferdinand von Helmholtz,[7] Harry Partch,[8] Reinier Plomp and Willem J. M.

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[1] GIRBAU i BADÓ, Joan: "Les Matemàtiques i les escales musicals". In: *Butlletí de la Secció de Matemàtiques de la Societat Catalana de Ciències Físiques, Químiques i Matemàtiques.* Vol. 18 (October 1985), pp. 3-27. Also available in: *Institut d'Estudis Catalans. Portal de Publicacions* [online]. In: http://publicacions.iec.cat/repository/pdf/00000011/00000017.pdf>. [December 2010]. A more informal draft of this article was published in GIRBAU i BADÓ, Joan: "Las matemáticas y las escalas musicales". In: *La Vanguardia*, 15 October 1988, p. 6.

[2] SÁNCHEZ GONZÁLEZ, Francisco Javier: *Francisco Javier Sánchez González* [online]. In: <http://usuarios.multimania.es/melpomenia/>. [May 2011]. Francisco Javier Sánchez González's Web site.

[3] GARCÍA SUÁREZ, Carlos: "Representación algebraica de las escalas musicales". In: *Música. Revista del Real Conservatorio Superior de Música de Madrid,* Vols. 10/11 (2003-2004), pp. 113-132.

[4] BLACKWOOD, Easley: *The structure of recognizable diatonic tunings.* Princeton, New Jersey: Princeton University Press, 1985. ISBN 0-691-09129-3.

[5] LINDLEY, Mark (with R. Turner-Smith): *Mathematical Models of Musical Scales: A New Approach.* Bonn: Verlag für systematische Musikwissenschaft GmbH, 1993. ISBN 3-922626-66-1

[6] SETHARES, William A. *Tuning, Timbre, Spectrum, Scale*. London: Springer-Verlag London Limited, 1998, 345 p. ISBN 3-540-76173-X. There are other preview works by the same author: SETHARES, William A. "Local consonance and the relationship between timbre and scale". In: *Journal of Acoustics Society of America*, Vol. 94, No. 3 (1993), pp. 1218-1228. SETHARES, William, A. "Adaptative tunings for musical scales". In: *Journal of Acoustical Society of America*, Vol. 96, No. 1 (1994), pp. 10-18. There is more information on the author's Web site: SETHARES, William A.: *William A. Sethares (Bill)* [online]. In: <http://eceserv0.ece.wisc.edu/~sethares/index.html>. [December 2010].

[7] HELMHOLTZ, Hermann Ludwig Ferdinand von: *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik*. Braunschweig: Friedrig Vieweg & Sohn, 1863/1870/1877/1896. Sixth edition: 1913, 668 pages. Reprints: Darmstadt: Wissenschaftliche Buchges,

Levelt[9] and provided the main theme for developing my Final Project in my Degree in Telecommunications Engineering.[10] As these theories show, tunings are closely related to the spectrum of the instruments. This is one of the justifications for the choice of this project for a degree in telecommunication engineering since spectra are one of the most important themes studied.[11] The software I programmed for the project was used to look for a solution to the problem of temperament in Johann Sebastian Bach's *Well-Tempered Clavier*. This part of the project was accepted for publication in a journal of musicology.[12]

The mathematical theories about tuning and temperament based on vectorial spaces were used in the development of the software and were partially set out in this work but they are not completely finished or drawn up yet.[13]

At the time of planning a new project to complete the first part of my doctoral studies, I thought of several propositions but finally decided that the best solution was to continue with the same theme, taking into account that lots of articles and new ideas had been published in reference to temperament in Bach's *Well-Tempered Clavier*. At least, if I could not find a new solution, this was a good occasion to evaluate the

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1968 (1913 edition); Frankfurt am Main: Minerva-Verlag, 1981 (edition of 1863), 600 pages. ISBN 3-8102-0715-2; Hildesheim: Georg Olms Verlag, 1983 (edition of 1968, 1913), 668 pages. Saarbrücken: VDM, Müller, 2007, 639 pages. ISBN 3-8364-0606-3. The 1863 edition is also available in: *The Virtual Laboratory Library* [online]. In: <http://vlp.mpiwg-berlin.mpg.de/library/data/lit3483?>. [December 2010]. Excerpt of the 1896 edition: *Die Homepage von Joachim Mohr. Startseite* [online], <http://delphi.zsg-rottenburg.de/rein_helmholtz.html> [December 2010]. English edition, translated by Alexander John Ellis with the author's approval from the 3rd German edition, with additional notes and an additional appendix: London: Longmans, Green, and Co., 1875, 824 pages. ISBN 1-85506-602-5. Second English edition, translated by Alexander John Ellis, based on the 4th German edition of 1877 with extensive notes, foreword and afterword: London: Longmans, Green, and Co., 1885. Fourth English edition: 1912, 575 pages. Reprint of the 1875 English edition: Bristol: Thoemmes Press & Tokyo: Maruzen, 1998. Reproduction of the second English edition of 1885 with a new introduction by Herny Margenau: New York: Dover Publications, Inc., 1954/1961, 576 pages. ISBN 0-486-60753-4. French edition, translated by G. Guéroult, based on the 1863 German edition: *Théorie physiologique de la musique fondée sur l'étude des sensations auditives*. Paris: Victor Masson & Fils, 1874, 544 pages. Reprint: Paris: Jacques Gabay, 1990..

[8] PARTCH, Harry. *Genesis of a music: An account of a creative work, its roots and its fulfillments*. New York: Da Capo Press, 1974, 517 p. ISBN 0-306-71597-X.

[9] PLOMP, Reinier: *Aspects of Tone Sensation*. London: Academic Press, 1976. PLOMP, Reinier & LEVELT, Willem J. M. "Tonal consonance and critical bandwidth". In: *Journal of Acoustical Society of America*, Vol. 38 (1965), pp. 548-560. The authors produced a preliminary work: "Musical consonance and Critical Bandwidth". In: *Fourth International Congress on Acoustics*. Copenhagen, 1962.

[10] MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software para el estudio de aplicaciones del análisis espectral en la musicología histórica y la etnomusicología*. Final Project for Telecommunications Engineering Degree. Barcelona: Universitat Politècnica de Catalunya (UPC), July 2003, 275 pp. Also available in: *Documentation - Sergio Martínez Ruiz* [online]. 2 March 2010. In: <http://sites.google.com/site/sergiomartinezruiz2/documentation> & *Documentación - Sergio Martínez Ruiz* [online]. 2 March 2010. In: <http://sites.google.com/site/sergiomartinezruiz1/documentacion>. [June 2011].

[11] More information on these theories is given in the works cited above, other works cited in the 'Bibliography' and also briefly set out in MARTÍNEZ RUIZ, Sergio: *Desarrollo de un sotware..., op. cit.*; and MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia y la afinación en la obra de *El clave bien temperado* de J. S. Bach. In: *Revista de Musicología*, Vol. 27, No. 2 (December 2004), pp. 895-931. Also available in: *Documentation - Sergio Martínez Ruiz* [online]. 13 March 2010. In: <http://sites.google.com/site/sergiomartinezruiz2/documentation> & *Documentación - Sergio Martínez Ruiz* [online]. 13 March 2010. In: <http://sites.google.com/site/sergiomartinezruiz1/documentacion>. [June 2011].

[12] MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia…", *op. cit.*

[13] They will be developed in: MARTÍNEZ RUIZ, Sergio: *A mathematical model for musical tunings and temperaments.* Under preparation.

temperaments created or chosen by these authors from the point of view of sensory dissonance. This was the main question evaluated by *SpecMusic*, the programme developed by myself for my Final Degree Project.[14]

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- My relatives and friends, for their support.

Introduction

The title of Bach's work *Well-Tempered Clavier* suggests that something in relation to temperament was taken into account at the time of its composition. We are not able to comment on this definitively. The study of the question can start from several temperaments which existed at the time of Bach and perhaps from several of Bach's contemporary writings none of which gives any specific temperament for the work.

Because of this, several suggestions were made during the second half of the twentieth century and, even moreso, during the first ten years of the twenty-first. They have been based on a variety of aspects and theories developed by the authors cited.

[14] This program is available on my Web site: MARTÍNEZ RUIZ, Sergio: *Sergio Martínez Ruiz* [online]. In: <http://sites.google.com/site/sergiomartinezruiz/>. [December 2010].
[15] BILLETER, Bernhard: "Zur "Wohltemperirten" Stimmung von Johann Sebastian Bach: Wie hat Bach seine Cembali gestimmt?". In: *Ars Organi*, Vol. 56/1 (March 2008), pp. 18-21.
[16] MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software..., op. cit.*
[17] MARTÍNEZ RUIZ, Sergio: *La teoría de la disonancia..., op. cit.*

However, none of these propositions can be taken as definitive since there is no evidence that Bach uses any of the temperaments suggested. It is only possible to arrive at an approximate solution, which can be adjusted to the aesthetics of Bach's work and the aesthetics of Baroque music in general.

Objectives

Starting from all known solutions to the problem of temperament in Johann Sebastian Bach's *Well-Tempered Clavier*, the objective of this work is to analyse all these given solutions according to a different point of view, without negating any of them.

The method of analysis used for this research is based on the mathematical theories of sensory dissonance, which have been developed by several authors: Hermann Ludwig Ferdinand von Helmholtz,[18] Harry Partch,[19] Reinier Plomp and Willem J. M. Levelt[20] and William A. Sethares.[21] All of these theories build on the previous ones. Sensory dissonance considers the intervals and chords themselves, independently of the context in which they are found. In a more objective way, sensory dissonance makes reference to the sensation that such intervals produce in our brain. Its definition is based on its physical and physiological causes, that is to say, the working of the basilar membrane in the inner ear, the timbre and the intensity of the sound and the relation of frequency between both notes which compound the interval.

A specific mathematical model for dissonance, which depends on the difference of frequencies of the interval and the spectrum of the sound is given by Sethares[22] as a generalization of the previous theories. As a consequence, sensory dissonance also depends on the instrument and the temperament in use for the piece and has to be analysed from this point of view.

According to this theory, a measurement of sensory dissonance can be given for a particular piece performed using a specific tuning or temperament and with a specific instrument. Depending on the tuning or temperament used, a different degree of dissonance will be obtained after a suitable analysis of the piece. It is easy to think that a temperament for which a low degree of dissonance is obtained for a concrete piece can be considered as a good temperament to be used in the performing of that piece.

This method will be applied to the *Well-Tempered Clavier* by Johann Sebastian Bach using the programme *SpecMusic*, a computing tool developed by myself which is capable of applying Sethares' mathematical model to a score.[23] After this evaluation has taken into account all known probable temperaments which could be used for the work, a new solution will be proposed from the point of view of sensory dissonance. The level of dissonance will be calculated for each piece of the work, applying several temperaments for the evaluation. A spectral model for a harpsichord will be considered for the analysis.

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[18] HELMHOLTZ, Ludwig Ferdinand von: *Die Lehre von den Tonempfindungen...*, *op. cit.*
[19] PARTCH, Harry: *Genesis of a music...*, *op. cit.*
[20] PLOMP, Reinier: *Aspects of Tone Sensation...*, *op. cit.*; PLOMP, Reinier & LEVELT, Willem J. M.: "Tonal consonance and critical bandwidth", *op. cit.*; PLOMP, Reinier & LEVELT, Willem J. M.: "Musical consonance and Critical Bandwidth"..., *op. cit.*
[21] SETHARES, William A.: *Tuning,…*, *op. cit.*
[22] SETHARES, William A.: *Tuning,…*, *op. cit.*
[23] This programme is available in my Web site: MARTÍNEZ RUIZ, Sergio: *Sergio Martínez Ruiz* [online]. In: <http://sites.google.com/site/sergiomartinezruiz/>. [December 2010].

A decorative scroll in Bach's manuscript

Before reviewing all known opinions and theories about the tuning of the harpsichord in the *Well-Tempered Clavier*, it is interesting to notice the existence of a decorative scroll in the title page of the autograph manuscript,[24] supposedly drawn by Bach himself.

Figure 1 - Scroll in the title page of the authograph manuscript of the *Well-Tempered Clavier* by Johann Sebastian Bach

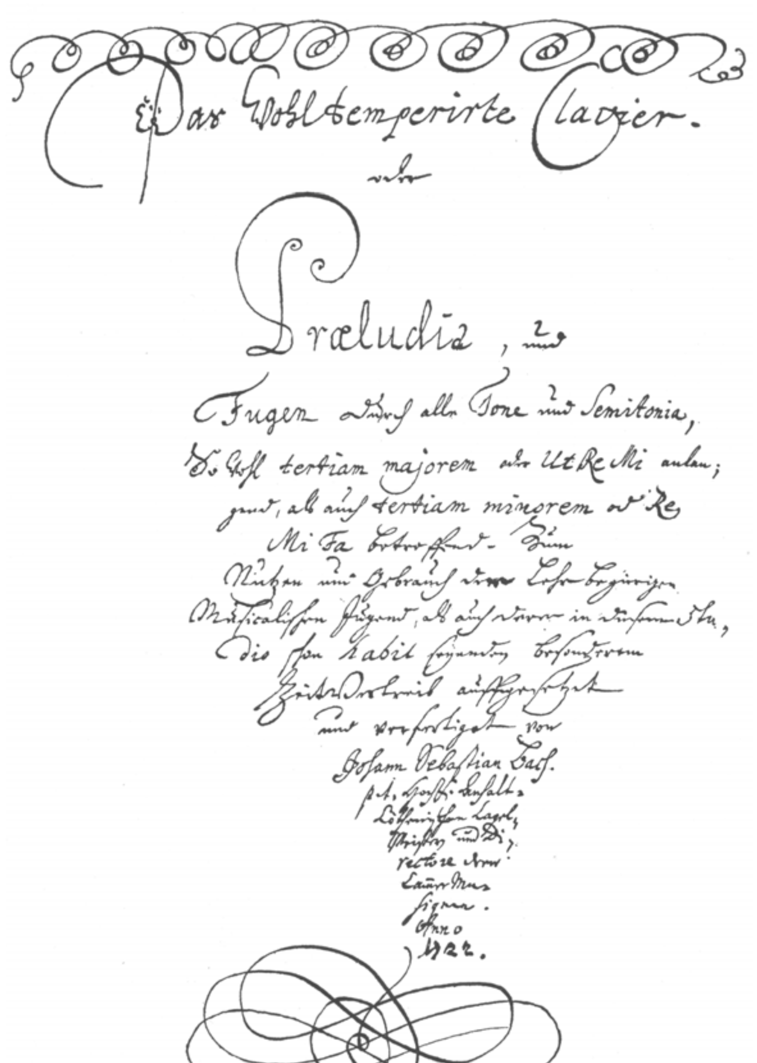

Figure 2 - Title page of the authograph manuscript of the *Well-Tempered Clavier* by Johann Sebastian Bach

[24] BACH, Johann Sebastian: *Das Wohltemperirte Clavier oder Prœludia, und Fugen durch alle Tone und Semitonia, so wohl tertiam majorem oder Ut Re Mi anlangend, als auch tertiam minorem oder Re Mi Fa betreffend. Zum Nutzen und Gebrauch der Lehrbegierigen Musicalischen Jugend, als auch derer in diesem studio schon habil seyenden besonderem Zeitvertreib auffgesetzet und verfertiget von Johann Sebastian Bach. p. t: Hochfürstlich Anhalt-Cöthenischen Capel-Meistern und Directore derer Camer Musiquen. Anno 1722.* D: B Mus. ms. Bach P 415. Bach's authographed score, dated 1722, now in the collection of the Musikabteilung der Staatsbibliothek zu Berlin PreuBischer Kulturbesitz.

Certain characteristics of this picture makes one think that it has some relation to the temperament, even that Bach “notated a specific method of keyboard tuning”.[25]

The scroll is made up of 12 loops some of which have internal squiggles. Specifically, five of them have a more elaborate squiggle, another three have a simple squiggle and the remaining three are simple loops, that is, without any internal squiggle.

Some ornaments are found in this scroll which can give some clues to the interpretation of its meaning. Various interpretations of these ornaments have been given in a number of published articles about temperament in Bach's *Well-Tempered Clavier.*[26]

A list of these ornaments is given below:

Figure 3 - Ornament in 'D' of 'Das'

- Ornament in ‘D’ of ‘Das’: it can be associated to the note Eb/D#.

Figure 4 - Ornament in 'C' of 'Clavier'

- Ornament in “C” of “Clavier”: it can be associated to the note C.

Figure 5 - Beginning of the scroll

- The beginning of the scroll reminds a simple loop.

Figure 6 - End of the scroll

- The end of the scroll reminds a double loop.

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[25] LEHMAN, Bradley: *Johann Sebastian Bach's tuning* [online]. In: <http://www-personal.umich.edu/~bpl/larips/> & <http://www.larips.com>. [December 2010]. Bradley Lehman’s Web site.

[26] They are discussed in the following section ‘A historical survey’.

A historical survey[27]

In this section, all known suggested solutions will be set out with a short summary of their principles and the temperament that they suggest. In case a specific temperament is suggested, a nomenclature for it will be created in order to refer to that temperament in the rest of this work. This name will be indicated in brackets beside the title.

In case an author (contemporary or not) makes more than one proposition for the temperament, a Roman numeral is attached to each one. The numeration for these temperaments is based on - for historical temperaments - the way they are traditionally referred to and - for newly proposed temperaments - on chronological order. Where the same author proposes more than one temperament in the same article or book, they are numbered following the original order.

The historical temperaments suggested in this survey were among those proposed by the following authors: Werckmeister, Kirnberger, Neidhardt, Tartini - Vallotti and Sorge.[28]

Carl Philipp Emanuel Bach (1753)[29]

It's easy to think that Carl Philipp Emanuel Bach could use the same temperament as his father's, or very close to it. Nevertheless, there is no evidence of this fact. The method of tuning of Johann Sebastian Bach and his son Johann Philipp Emanuel is also unkown.

Carl Philipp Emanuel's own work makes reference to the method of tuning of his father but it doesn't describe the general strategy of tempering "most of" the fifths, and they describe its effect in musical practice. Two important excerpts of this work are reproduced bellow. The first passage brings out the distinction that this is the new correct way to do things, as opposed to the older regular (*id est*, "menatone") schemes:

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[27] Other surveys of 'Bach' temperaments can be found in: LEHMAN, Bradley: "Other 'Bach' temperaments". In: *Johann Sebastian Bach's tuning* [online]. 2005-10. In: <http://www-personal.umich.edu/~bpl/larips/bachtemps.html> & <http://www.larips.com>. [December 2010]; DI VEROLI, Claudio: *Unequal Temperaments. Theory, History and Practice. Scales, tuning and intonation in musical performance.* First edition: Bray (Ireland): Bray Baroque, November 2008. Second revised edition: Bray (Ireland): Bray Baroque, April 2009, 456 pages. eBook available in: *Soluciones de auto publicación e impresión de libros - Libros, eBooks, álbumes de fotos y calendarios en Lulu.com* [online]. In: <http://www.lulu.com/product/ebook/unequal-temperaments-theory-history-and-practice/4741955>. [January 2011]. Previous edition: *Unequal Temperaments and their Role in the Performance of Early Music. Historical and theoretical analysis, new tuning and fretting methods*. Buenos Aires (Argentina): Artes Gráficas Farro, 1978, 326 pages. Section 9.7, pp. 127-135.

[28] A brief explanation of these temperaments can be found in 'Appendix 4: Description of some historical 'good' temperaments'. For more information, several references are given in the 'Bibliography'.

[29] BACH, Carl Philipp Emanuel: *Versuch über die wahre Art das Clavier zu spielen*. Berlin, 1753 (Teil 1) & 1762 (Teil 2), and several later editions. Reprint by Lothar Hoffmann-Erbrecht: Leipzig: Breitkopf & Härtel, 1958/1992. English translation by William J. Mitchell: *Essay on the True Art of Playing Keyboard Instruments*. New York: Norton, 1949; BACH, Carl Philipp Emanuel: Letter to Forkel. Hamburg, 1774. Published in: SCHULZE, Hans-Joachim, MAUL, Michael and WOLFF, Christoph (eds.): *Biographische Mittelungen über Johann Sebastian Bach.* In: *Dokumente zum Nachwirken Johann Sebastian Bachs 1750-1800,* Vol. 3. Kassel and Leipzig, 1972, p. 285. Reedition: *Johann Nicolaus Forkels Bach-Biographie (1802) und zugehörige Materialien. Bach-Dokumente.* In: *Die Neue Bach-Ausgabe*, Supplement, Vol. 8. Kassel & Leipzig: Bärenreiter, 2008. *Vid.* LEHMAN, Bradley: "Carl Philipp Emanuel Bach". In: *Johann Sebastian Bach's tuning* [online]. In: <http://www-personal.umich.edu/~bpl/larips/> & <http://www.larips.com>. [December 2010]. Bradley Lehman's Web site.

"§. 14. Beyde Arten von Instrumenten müssen gut temperirt seyn, indem man durch die Stimmung der Quinten, Quarten, Probirung der kleinen und grossen Tertien und gantzer Accorde, den meisten Quinten besonders so viel von ihrer größten Reinigkeit abnimmt, daß es das Gehör kaum mercket und man alle vier und zwantzig Ton=Arten gut bracuhen kan. Durch Probirung der Quarten hat man den Vortheil, daß man die nöthige Schwebung der Quinten deutlicher hören kan, weil die Quarten ihrem Grund=Tone näher liegen als die Quinten. Sind die Claviere so gestimmt, so kan man sie wegen der Ausübung mit Recht für die reinste Instrumente unter allen ausgeben, indem zwar einige reiner gestimmt aber nicht gespielet werden. Auf dem Claviere spielet man aus allen vier und zwantzig Ton=Arten gleich rein und welches wohl zu mercken vollstimmig, ohngeachtet die Harmonie wegen der Verhätnisse die geringste Unreinigkeit sogleich entdecket. Durch diese neue Art zu temperiren sind wir weiter gekommen als vor dem, obschon die alte Temperatur so beschaffen war, daß einige Ton=Arten reiner waren als man noch jetzo bey vielen Instrumenten antrift. Bey manchem andern Musico würde man vielleicht die Unreinigkeit eher vermercken, ohne einen Klang=Messer dabey nöthig zu haben, wenn man die hervorgebrachten melodischen Töne harmonisch hören sollte. Diese Melodie betrügt uns oft und läßt uns nicht eher ihre unreinen Töne verspüren, bis diese Unreinigkeit so groß ist, als kaum bey manchem schlecht gestimmten Claviere.[30]

§. 14. Both types of instrument must be tempered as follows: In tuning the fifths and fourths, testing minor and major thirds and chords, take away from most of the fifths a barely noticeable amount of their absolute purity. All twenty-four tonalities will thus become usable. The beats of fifths can be more easily heard by probing fourths, an advantage that stems from the fact that the tones of the latter lie closer together than fifths. In practice, a keyboard so tuned is the purest of all instruments, for others may be more purely tuned but they cannot be purely played. The keyboard plays equally in tune in all twenty-four tonalities and, mark well, with full chords, notwithstanding that these, because of their ratios, reveal a very slight impurity. The new method of tuning marks a great advance over the old, even though the latter was of such a nature that a few tonalities were purer than those of many present non-keyboard instruments, the impurity of which would be easier to detect (and without a monochord) by listening harmonically to each melodic tone. Their melodies often deceive us and do not expose their impurity until it is greater than that of a badly tuned keyboard.[31]

He also said of his father: “No one could tune and quill his instrument to his satisfaction”.[32]

For the hypothesis that Carl Philipp Emanuel Bach was describing the same temperament his father used, the stronger corroboration is done here by playing through Car Philipp Emanuel Bach’s own music, and the other music he knew at court. This includes his father’s Musical Offering, where the ricercars work unproblematically in this tuning but sound rotten in the other typical unequal temperaments.

Regarding the Equal Temperament, if Carl Philipp Emanuel Bach had made reference to it (or something aurally indistinguishable from it), he would have said ”all of the fifths” instead of only “most of the fifths” get tempering.

Regarding the purity of the intervals, he said thet keyboards are “the purest of all instruments, for others may be more purely tuned but they cannot be purely played”. According to Lehman, this means that “keyboards have fixed intonation for each note, as opposed to the variable intonation of winds, strings, voice, etc… The other instruments and the voice might be “more purely tuned” on occasion for particular intervals, as in being closer to beatless pure intervals. But other parts of the musical

[30] BACH, Carl Philipp Emanuel: *Versuch über die wahre Art..., op. cit. Vid.* LEHMAN, Bradley: “Carl Philipp Emanuel Bach”..., *op. cit.*

[31] BACH, Carl Philipp Emanuel: *Versuch über die wahre Art..., op. cit.* English translation by William J Mitchell: *Essay on the True Art..., op. cit. Vid.* LEHMAN, Bradley: “Carl Philipp Emanuel Bach”..., *op. cit.*

[32] BACH, Carl Philipp Emanuel: Letter to Forkel. Hamburg, 1774..., *op. cit.*

fabric come into conflict… or the whole ensemble drifts up or down in pitch, as different parts are adjusted to one another to make the momentary intervals sound more nearly pure. The fixed-pitch nature of keyboards prevodes a stability against all this."

"The keyboard plays equally in tune in all twenty-four tonalities…". Likewise, the expression "in tune" means that all tonalities are "equally usable all the time, equally "pure" in sound and flexibility", but the keys do not all sound identical.

It is known that Carl Philipp Emanuel Bach's career temperament was the ordinary temperament at the royal court during his tenure there and this is confirmed by Johann Joachim Quantz in his flute treatise.[33] Quantz himself worked directly with at least three former pupils of Johann Sebastian Bach. Johann Friedrich Agricola, Christoph Nichelmann and Carl Philipp Emanuel Bach. Thus he had the opportunity to learn and play with the Bach temperament. Although he doesn't give any exact reference either, he says that using Bach's tuning, the dissonant and consonant content of the harmony matches his dynamic scheme, with regard to the treatment of tension, resolution and surprise. In more equalized temperaments these directional tendencies in the sound are minimized and can seem. Moreover, to teach the keyboard accompanist how to play dynamics expressively be the harmonic tendencies of dissonances and consonance, listening to the sound of every moment.

Rge relationship between dynamic and temnperament is also described by Carl Philip Emanuel Bach in his own treatise:

(...) Damit man alle Arten vom pianißimo biß zum fortißimo deutlich zu hören kriege, so muß man das Clavier etwas ernsthaft mit einiger Kraft, nur nicht dreschend angreiffen; man muß gegentheils auch nicht zu heuchlerisch darüber wegfahren. Es ist nicht wohl möglich, die Fälle zu bestimmen, wo forte oder piano statt hat, weil auch die besten Regeln eben so viele Ausnahmen leiden als sie festsetzen; die besondere Würckung dieses Schatten und Lichts hängt von den Gedancken, von der Verbindung der Gedancken, und überhaupt von dem Componisten ab, welcher eben so wohl mit Ursache das Forte da anbringen kan, wo ein andermahl piano gewesen ist, und offt einen Gedancken sammt seinen Con= und Dissonanzen einmahl forte und das andere mahl piano bezeichnet. Deswegen pflegt man gerne die wiederhohlten Gedancken, sie mögen in eben derjenigen Modulation oder einer andern, zumahl wenn sie mit verschiednen Harmonien begleitet werden, wiederum erschienen, durch forte und piano zu unterschieden. Indessen kan man mercken, daß die Dissonanzen insgemein stärcker und die

(...) In order to control all shades from pianissimo to fortissimo the keys must be gripped firmly and with strength. However, they must not be flogged; but on the other hand there must not be too much restraint. It is not possible to describe the contexts appropriate to the forte or piano because for every case covered by even the best rule there will be an exception. The particular effect of these shadings depends on the passage, its context, and the composer, who may introduce either a forte or a piano at a given place for equally convincing reasons. In fact, composite passages, including their consonances and dissonances, may be marked first forte and, later, piano. This is a customary procedure with both repetitions and sequences, particularly when the accompaniment is modified. **But in general it can be said that dissonances are played loudly and consonances softly, since the former rouse our emotions and the latter quiet them** *(exercise a)*. An exceptional turn of a melody which is designed to create a violent affect must be played loudly. So-called

[33] QUANTZ, Johann Joachim: *Versuch einer Anweisung die Flöte traversiere zu spielen*. Berlin, 1752. Re-editions: Arnold Schering (ed.): Leipzig: C.F. Kahnt Nachfolger, 1906. xiv, 277 pp.; Leipzig, 1926. xiv, 277, 3pp. Facsimile edition by Hans-Peter Schmitz & Horst Augsbach (eds.): Kassel [u.a.]: Bärenreiter, 1953, 7 p.l., 334, [20], 24, [2]p; Reprints: *Documenta Musicologica*, Vol. 1, No. 2 (1953), 334, [20], 24, [2] pp.; 1983. ix, 3, 419 pp. ISBN: 3761807112; VEB Deutscher Verlag für Musik, c1983; München: Deutscher Taschenbuch Verlag, 1983; Kassel [u.a.]: Bärenreiter, 1992. x, 424 pp. English translation: Edward R. Reilly (ed.): *On Playing the Flute*. London: Faber & Faber, 1966. 365 pp.; 1976. xxxix, 368 pp. ISBN: 0571110339. Second English edition: London: Faber, 1985. xliii, 412 pp. ISBN: 0571180469; New York: Schirmer Books, 1985. xliii, 412p. ISBN: 0028701607. Italian translation. S. Balestracci (ed.): *Trattato sul flauto traverso.* Lucca: Libreria Musicale Italiana Editrice, 1992.

[34] BACH, Carl Philipp Emanuel: *Versuch über die wahre Art..., op. cit.,* "performance" chapter *Vom Vortrage*. *Vid.* LEHMAN, Bradley: "Carl Philipp Emanuel Bach"..., *op. cit.*

Consonanzen schwächer gespielt werden, weil jene die Leidenschafften mit Nachdruck eheben und diese solche beruhigen (a). Ein besonderer Schwung der Gedancken, welcher einen hefftigen Affeckt erregen soll, muß starck ausgedruckt werden. Die so genannten Betrügereyen spielt man dahero, weil sie offt deswegen angebracht werden, gemeiniglich forte (b). Man kan allenfalls auch diese Regel mercken, welche nicht ohne Grund ist, daß die Töne eines Gesangs, welche ausser der Leiter ihrer Ton-Art sind, gerne das forte vertragen, ohne Absicht, ob es Con- oder Dissonanzen sind, und das gegentheils die Töne, welche in der Leiter ihrer modulirenden Ton-Art stehen, gerne piano gespielt werden, sie mögen consoniren oder dissoniren (c).

(...) Spielt man diese Probe=Stücke auf einem Flügel mit mehr als einem Griffbrette [Tastatur], so bleibt man mit dem forte und piano, welches bey einzeln Noten vorkommt, auf demselben; man wechselt hierinnen nicht eher, als biß gantze Passagien sich durch forte und piano unterscheiden. Auf dem Clavicorde fällt diese Unbequemlichkeit weg, indem man hierauf alle Arten des forte und piano so deutlich und reine heraus bringen kan, als kaum auf manchem andern Instrumente. Bey starcker oder lärmender Begleitung muß man allezeit die Haupt=Melodie durch einen stärckern Anschlag hervorragen lassen.[34]

deceptive progressions are also brought out markedly to complement their function *(b)*. **A noteworthy rule which is not without foundation is that all tones of a melody which lie outside the key may well be emphasized regardless of whether they form consonances or dissonances and those which lie within the key may be effectively performed piano, again regardless of their consonance or dissonance** *(c)*.

(...) If the Lessons are played on a harpsichord with two manuals, only one manual should be used to play detailed changes of forte and piano. It is only when entire passages are differentiated by contrasting shades that a transfer may be made. This problem does not exist at the clavichord, for on it all varieties of loud and soft can be expressed with an almost unrivaled clarity and purity. A loud, boisterous accompaniment must always be balanced by a stronger melodic touch.[35]

Robert Halford Macdowall Bosanquet (1876)[36]

Robert Halford Macdowall Bosanquet, in 1876, already questioned the use of the equal temperament by Bach although he doesn't give any specific solution to this question.

He stated that the clavichord was the preferred instrument for Bach: "It appears that Bach possessed a clavichord and a harpsichord," nevertheless, "Bach's favourite instrument was the clavichord. He considered it the best instrument for the house, and for study." Moreover, "it appears that Bach's clavier compositions were regarded both by himself and others as specially dedicated to the clavichord."

Although "now it is occasionally said that 'Bach preferred the equal temperament' [...], so far as Bach's clavier music is concerned therefore, the appeal to his authority in favour of the equal temperament falls to the ground. The argument is unfounded in other respects. Bach compared the equal temperament with the defective

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35 BACH, Carl Philipp Emanuel: *Versuch über die wahre Art..., op. cit.* English translation by William J Mitchell: *Essay on the True Art..., op. cit. Vid.* LEHMAN, Bradley: "Carl Philipp Emanuel Bach"..., *op. cit.* Emphases by Bradley Lehman.

36 BOSANQUET, Robert Halford Macdowall: *An Elementary Treatise on Musical Intervals and Temperament, with an account of an enharmonic harmonium exhibited in the loan collection of scientific instruments, South Kensington, 1876, also of an enharmonic organ exhibited to the Musical Association of London, May, 1875.* London: MacMillan & Co., 1876. 2nd edition with introduction by Rudolf Rasch (ed.): Tuning and temperament library, Vol. 4. Utrecht: Diapason Press, 1987, 179 pages. Several reprints: Diapason, 1987, 94 pages, ISBN 9070907127; Kessinger Publishing's Rare Reprints, 2008, 116 pages, ISBN 1436889138; Cornell University Library-Published 2009, 126 pages, ISBN 1112579346; General Books LLC, 2010, 58 pages, ISBN 1152183850; Nabu Press, 2010, 120 pages, ISBN 1176336231, p. 27-30.

mean-tone system on the ordinary keyboard, and with nothing else. His objection was to the wolf, and cannot be counted as of force against arrangements in which the wolf does not exist."

James Murray Barbour (1947)[37]

James Murray Barbour, in 1947, questioned that the title *Well-Tempered Clavier* made reference to the Equal Temperament again. As Barbour says, "even in Bach's day there was a good German phrase for Equal Temperament: "*die gleichschwebende Temperatur*", that is to say, "the equally beating temperament"".[38] Although the expression "well-tuned" had been used by Stevin and Rameau with a meaning similar to Equal Temperament, it was certain that this expression had a very different meaning for German theorists contemporaries to Bach. German "good" temperaments referred to the fact that all keys could be played without offense to the ear".[39] We can also make reference to some words by Werckmeister himself: "But if we have a well-tuned clavier, we can play both the major and minor modes on every note and transpose them at will. To one who is familiar with the entire range of keys, this affords variety upon the clavier and falls upon the ear very pleasantly."[40] Evidently, all these affirmations refers to the possibility of using all tonalities of the new tonal system at that time.

Herbert Kelletat (1960)[41] *[Kirnberger III & Kelletat]*

Herbert Kelletat has suggested that one of the temperaments published by Kirnberger[42] was in fact Bach's own.[43] He also made his own formulation of a 'Bach' temperament giving strong credence to Kirnberger.

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[37] BARBOUR, James Murray: "Bach and the 'Art of Temperament'", In: *Musical Quarterly*, Vol. 33, nº. 1 (January of 1947), pp. 64-89. Also available in *Garland Library of HWM*, Vol. 6 (1985), pp. 2-27.

[38] BARBOUR, James Murray: *Tuning and Temperament: A Historical Survey.* East Lansing: Michigan State College Press, 1951. Reprint in New York: Da Capo Press, 1973. Reprint in New York: Dover, 2004. 228 pages. ISBN 0-486-43406-0, p. 194.

[39] MATTHESON, Johann. *Critica musica*. Hamburg, 1722-1725. Reprint: 1964; Laaber: Laaber Verlag, 2000, 2 vols, 388/416 pages, Vol. II, p. 162. *Vid.* BARBOUR, J. Murray: *Tuning and temperament…, op. cit.*, p. 194.

[40] WERCKMEISTER, Andreas: *Musicalische Temperatur, Oder deutlicher und warer Mathematischer Unterricht / Wie man durch Anweisung des Monochordi ein Clavier / sonderlich die Orgelwerke / Positive, Regale, Spinetten / und dergleichen wol temperirt stimen könne / damit nach heutiger manier alle Modificti in einer angenehm- und erträglichen Harmonia mögen genommen werden / Mit vorhergehender Abhandlung ... der Musicalischen Zahlen / ... Welche bey Einrichtung der Temperaturen wohl in acht zu nehmen sind*. Quedlinburg (s. n.), 1691. Reprints: RASCH, Rudolf A. (ed.): Utrecht: Diapason Press, 1983; PFEIFFER, Rüdiger (ed.): Essen: Die Blaue Eule, 1996, 138 pages. English translation by Elizabeth Hehr: Oberlin: Oberlin University, thesis paper. *Vid.* BARBOUR, J. Murray: *Tuning and temperament…, op. cit.*, p. 195.

[41] KELLETAT, Herbert: *Zur Musikalischen Temperatur insbesondere bei Johann Sebastian Bach.* Kassel: Onckel Verlag, 1960. Reprint 1979, 96 pp. 2nd edition: *Zur musikalischen Temperatur. Deel I. Johann Sebastian Bach und seine Zeit.* Kassel: Verlag Merseburger, 1981. 100 pp. ISBN 3-87537-156-9. *Vid.* also KELLETAT, Herbert: "Zur Frage der Tonordnung bei der Restaurierung alter Orgeln". In: *The Organ Yearbook,* Vol. 8 (1977), pp. 61-63.

[42] KIRNBERGER, Johann Phillip: *Die Kunst des reinen Satzes in der Musik (aus sicheren Grundsätzen hergeleitet und mit deutlichen Beyspielen erläutert)*. I. Theil. Berlin & Königsberg: Heinrich August Rottmann, 1774, 250 pages. II. Theil. Berlin & Königsberg: G.J. Decker und G.L. Hartung, 1776-1779, Wien, 1793, 252 pages. Reprint: Hildesheim: Georg Olms Verlag, 1968 & 2010, 823 pages.

[43] *Vid.* BARNES, John: "Bach's keyboard temperament: Internal evidence from the *Well-Tempered Clavier*". In: *Early Music*, Vol. 7, No. 2 (April 1979), pp. 236-249.

Kirnberger was one of Bach's pupils and he affirmed that he has been received the knowledge of his master. His third temperament is described by him in 1779 in a letter to Forkel.[44] According to these statements, Kelletat derived his own proposition from Kirnberger's third temperament and he presented it in this way:

"[...] An empirical well-tempering on the diatonic and chromatic basis of classical meantone temperament. It follows the foundations of the tonal system laid by Bach's pupil Kirnberger and is confirmed amongst other things by the interval analysis of Bach's *Well-Tempered Clavier*".[45]

According to Dominique Devie,[46] this affirmation is not completely clear. She thinks that it is "a possible option but not exclusive". The sentence making reference to the teaching of the harmonic system but not of the temperaments. Moreover, Kinrberger's declarations[47] don't let to stablish this heritage from Bach.

Kelletat's temperament really consists in a blend of Werkmeister III and Kirnberger III temperaments and it is designed reducing the fifths C-G-D-A by 1/4 of a Pythagorean comma.[48] These fifths are those which are already reduced by the same amount both in Kirnberger III and Werckmeister III temperaments. In order to avoid having the C-E major third smaller than pure, the fifth A-E is reduced by 1/9 of a Pythagorean comma, that is, slightly less tempered than it is done in the Kirnberger III temperament. The remaining amount, that is, 1/12 of a Pythagorean comma, is placed into the fifth C-F. The resulting layout can be shown in the following figure:

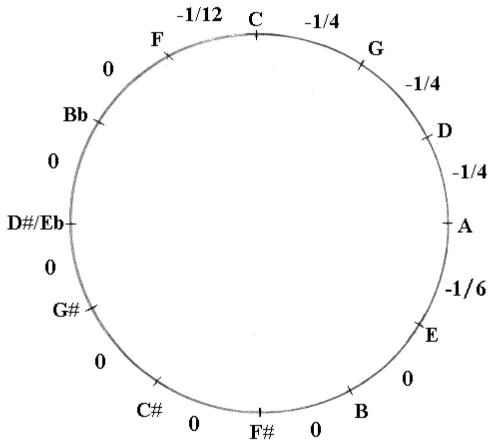

Figure 7 - Kelletat temperament

[44] KIRNBERGER, Johann Phillip: Letter to Forkel, [1779]. Published in: BELLERMAN, Heinrich: „Briefe von Kirnberger an Forkel". In: *Allgemeine musikalische Zeitung* ("General music journal"), No. 34 (1871). Transcription in "The letter from Johann Philipp Kirnberger to Johann Nikolaus Forkel, in which the former defines the keyboard temperament that is now known as 'Kirnberger III'". In: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/Kirn_1871.html>. [January 2011]. More information in: "Johann Philipp Kirnberger's definition of the temperament that is now known as 'Kirnberger III'". In: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/K_III.html>. [January 2011].

[45] KELLETAT, Herbert: *Ein Beitrag zur musikalischen Temperatur der Musikinstrumente vom Mittelalter bis zur Gegenwart*. Reutlingen: Wandel und Goltermann, 1966, p. 26.

[46] DEVIE, Dominique: *Le tempérament musical. Philosophie, histoire, théorie et pratique.* Béziers: Société de musicologie du Languedoc, 1990, 540 pages. ISBN 2-905400-52-8. Reprint and extracts in: *Reprints - Fac-simile - Musique ancienne de clavier* [online]. In: <http://musicreprints.free.fr/tempmus/pdf/tm_rfsm.pdf>. [August 2010], 18 pages.

[47] KIRNBERGER, Johann Phillip: *Die Kunst des reinen Satzes..., op. cit.*

[48] *Vid.* Section "Appendix 4: Description of some historical 'good' temperaments" for the explanations of the temperaments of Kirnberger and Werckmeister.

Herbert Anton Kellner (1977)[49] *[Kellner]*

Herbert Anton Kellner proposed a new 'Bach' temperament according to the study of the spirit of the Baroque principles of music and also vased on the calculation of the bats rates of the intervals.

The intervals of fifth, major third and minor third are identified in Kellner's articlele as *Q*, *T* and *R* respectively. The tempered versions of these intervals are identified as Q_d, T_d and R_d respectively. The calculation of the beat rates for each of these intervals is:

a) For the fifth, *Q=3/2*, the third partial of the lower note coincides with the second partial of the upper note, that is:

$$3*1 = 2*Q$$

b) For the major third, *T=5/4*, the fifth partial of the lower note coincides with the fourth partial of the upper one, that is:

$$5*1 = 4*T$$

c) For the minor third, *R=6/5*, the sixth partial of the lower note coincides with the fifth partial of the upper one, that is:

$$6*1 = 5*R$$

For tempered intervals, that is, intervals which are not exactly pure, but slightly out of tune, the relevant upper partials no longer coincides; instead, their small differences in frequency cause beats. The bat-rates can be calculated with the following expressions related to the upper closer partials:

$$S(Q_d) = 2*Q_d - 3$$
$$S(T_d) = 4*T_d - 5$$
$$S(R_d) = 5*R_d - 6$$

In the Baroque period, the triad was regarded as a musical symbol of the Trinity. The major triad was designated as *trias harmonica perfecta* and the minor triad as *trias harmonica imperfecta*, since the major triad is closer to the *unitas* unity than the minor triad. This is due to their ratios: 1:5/4:3/2 (4:5:6 in integers) and 1:6/5:3/2 (10:12:15 in integers). "The *Trias Harmonica* is a mutual connection between Third and Fifth set to a fundamental note which can be done in all tonalities, and both in major and minor [...] and is otherwise called *Radix Harmonica*, because all Harmony derives therefrom".[50]

In a well-tempered triad, the fifth is to be flattened and the third sharpened. As the perfection is symbolized by the *unitas* and rests in the ratio 1:1, it can be stipulated that the beats of this fifth Q_w to occur at the same rate as the beats of the well-tempered third T_w:

$$S(Q_w) = -S(T_w)$$
$$2Q_w - 3 = -(4T_w - 5)$$

Another equation can be obtained if the relation between both intervals is cosidered:

[49] KELLNER, Herbert Anton. "Eine Rekonstruktion der wohltemperierten Stimmung von Johann Sebastian Bach". In: *Das Musikinstrument*, Vol. 26, No. 1 (1977), pp. 34-35. English version: "A mathematical approach reconstructing J. S. Bach keyboard temperament". In: *Bach, the quarterly journal of the Riemenschneider Bach Institute*, Vol. 10, No. 4 (October 1979), pp. 2-8, 22. Reprint: *Bach, the quarterly journal of the Riemenschneider Bach Institute*, Vol. 30, No 1 (Spring-Summer 1999), pp. 1-9.

[50] Johann Sebastian Bach's "Instructions on Figured Bass". *Vid.* SPITTA, Philipp: *Johann Sebastian Bach*. 2 Vols. Leipzig: Breitkopf & Härtel, 1873-1880 (1962). ISBN: 978-3-7651-0921-8. Abridged version with notes by W. Schmieder: Leipzig: Breitkopf & Härtel, 1935 (1954). English translation: 1884-1899 (1952), p. 917.

$$T_w = Q_w^{\ 4}/4$$

$$2Q_w - 3 = -(4Q_w^{\ 4}/4 - 5)$$

$$Q_w^{\ 4} + 2Q_w - 8 = 0$$

which is called "Bach-equation".

A numerical resolution of the equation, based on an iteration method, gives the following results:

$$Q_w = 1.495953506$$

$$T_w = 1.252023247$$

Now, in the triad $1{:}T_w{:}Q_w$, the sharpened major third T_w beats from above at the same rate as the flattened fifth Q_w beats from below.

A small offset between the perfect the perfect fifth and the tempered one can be defined as:

$$B = Q/Q_w$$

which is called "Bach-*comma*".

Other possible approach to establish the well-tempered triad is to consider the minor triad and stipulate that the major third –now the upper interval- should beat at the same rate as the well-tempered fifth q_w associated with this triad. In this case the expression for the beats of the major third must be multiplied by the ratio q_w/t_w of the minor triad, before the beat-rates can be equated:

$$2q_w - 3 = -(q_w/t_w)*(4t_w - 5)$$

$$t_w = q_w^{\ 4}/4$$

$$2q_w - 3 = -(\frac{q_w}{q_w^{\ 4}/4})*(\frac{4q_w^{\ 4}}{4} - 5)$$

$$6q_w^{\ 4} - 3q_w^{\ 3} - 20 = 0$$

Solving numerically, the following results for q_w and t_w intervals can be obtained:

$$\mathrm{q_w} = 1.495865822$$

$$\mathrm{t_w} = 1.251729726$$

A new offset can be defined for this case for the relation between the perfect fifth and the tempered one:

$$\beta = Q/q_w$$

To tuning the whole temperament, the twelve fifths of the cycle have to be considered. A set of *n* tempered fifths and *m* pure fifths can be combined in order to get the octave (more exactly, the value of 7 octaves):

$$Q_w^{\ n}Q^m = 2^7 = 128$$

It has been taken into account that *n=m=12*, then:

$$Q_w^{\ n}Q^m = 2^{12-n} = 128$$

The value of *n* that yields the most approximated result to the equation is *n=5*, for both cases. Thus the entire tuning-system is characterized by seven pure fifths and five flattened ones.

Regarding the Baroque concept of the *unitas*, the well-tempered triad – distinguished by its perfection- must not occur than just once within the tuning-system, symbolizing the Tri-Unity. This well-tempered triad, most naturally, is placed upon C major, the centre of tonality.

Then, the tempered fifths can be distributed in the following way: four successive fifths starting at C and finishing at E, third of the C-major triad, and the other fifth between B and F#, isolated from the rest to avoid two well-tempered triads. The rest of the intervals are pure fifths. The result can be shown in the following figure:

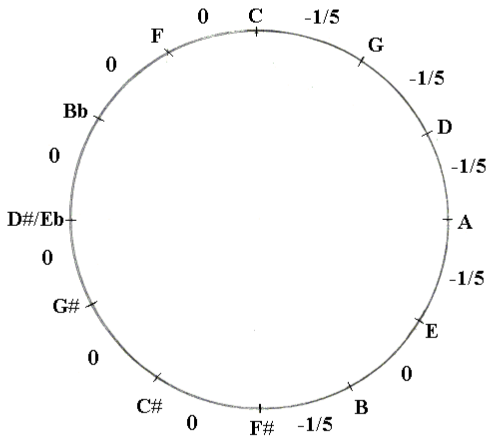

Figure 8 - Kellner temperament

The characteristics of tonalities show a typical graduation in the Baroque fashion. The quality of major thirds is determined by the purity of their major thirds and the minor ones, by their minor thirds. Three major keys have Pythagorean tonic triads. This temperament can be easily implemented by ear and "exceedingly accurately."

Bernhard Billeter (1977)[51] [Billeter I]

Bernhard Billeter presented a temperament whose conjectures are based on a Gottfried Silbermann temperament which is also presented in his book. The structure of his new temperament is very similar to Kirnberger II temperament. Most of the *syntonic* comma is placed into the fifths D-A-E, as in Kirnberger temperament. Instead of 1/2 of a *syntonic* comma into D-A-E each, as happens in Kirnberger temperament, Billeter used 1/3 of a *syntonic* comma. The remainder is spread among the fifths on either side: F-C-G-D and E-B-F#, including the *schisma*. The resulted layout is shown in the following figure:

[51] BILLETER, Otto Bernhard: "Anweisung zum Stimmen von Tasteninstrumenten in verschiedenen Temperaturen". In: *Österreichische Musikzeitschrift*, Vol. 32, No. 4 (1977), pp. 185-195. *Vid.* also BILLETER, Otto Bernhard: *Anweisung zum Stimmen von Tasteninstrumenten in verschiedenen Temperaturen.* Veröffentlichung der Gesellschaft der Orgelfreunde, Vol. 59. Berlin, Kassel: Verlag Merseburger, 1979, 1982. 40 pages, pp. 30-31and *Anhang*.

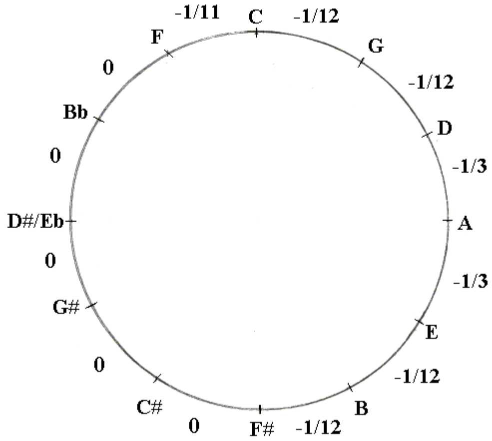

Figure 9 - Billeter I temperament

Nevertheless, his temperament is expressed in *schismata* according to the deviation of thirds. Its original definition is found in the following figure:[52]

```
        C#     G#     D#     A#
       5.5     9      10     11
      A      E      B      F#
     4.5     2      2      2
    F      C      G      D
   11     10      9      8
  Db     Ab     Eb     Bb
```

Figure 10 - Billeter I temperament (*schismata*)

John Barnes (1979, 1)[53] [Werckmeister III]

John Barnes, using a statitical analysis, justified the used of Werckmeister III[54] temperament (or other similar temperament[55]) in the *Well-Tempered Clavier* by Johann Sebastian Bach. His demonstration is based in the study of the dissonance degree for all thirds in the twelve fugues in major keys in the first volume.[56]

Barnes starts from the principle that more dissonant thirds would have to be used less often than those more consonant. Moreover, dissonant thirds should be used in special conditions for which the special characteristics of these intervals are not too much evident. For this thing, he evaluated all major thirds of all preludes of both parts of the work written in major tonalities. The evaluation consist to ascribe a prominence value for each occurrence of the interval depending on the perception of the possible

52 Its circular layout is set out by Bradley Lehman. *Vid.* LEHMAN, Bradley: "Other 'Bach' temperaments"..., *op. cit.*

53 BARNES, John: "Bach's keyboard temperament: Internal evidence from the *Well-Tempered Clavier*". In: *Early Music*, Vol. 7, No. 2 (April 1979), pp. 236-249.

54 WERCKMEISTER, Andreas: *Musikalische Temperatur..., op. cit.* This temperament already appeared in WERCKMEISTER, Andreas: *Erweiterte und verbesserte Orgel-Probe.* Quedlinburg, 1698. Aschersleben, 1716. Leipzig, 1754. Augsburg, 1783. Reprints: Kassel: Bärenreiter, 1927, 1970; Hildesheim: Georg Olms Verlag, 1972. English translation by Gerhard Krapf: *Werckmeister's Erweiterte und verbesserte Orgel-Probe in English*. Raleigh NC: Sunbury Press, 1976. Dutch translation by Jacob W. Lustig: *Orgelproef.* Amsterdam, 1755. Re-edition of the Dutch translation: Baarn: A. Bouman, 1968. French translation by Hippolyte Cellier: *L'orgelprobe de Werckmeister.* In: *L'orgue,* Vols. 98, 101 & 102 (1961-1962), pp. 48-50, 19-20, 34-37.

55 *Vid*. the next section "John Barnes (1979, 2) [Barnes]".

56 There is a more detailed explanation of Barnes' method in MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia...", *op. cit.*; and MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software...*, *op. cit.*

deviation in tuning respect just intonation. The prominence values run from A to E and they indicate how any tuning error is perceived depending on the context:

A: Very obvious.
B: Fairly obvious.
C. Easily perceptible.
D: Barely perceptible
E: Imperceptible

This method assumes that all intervals are exactly tuned according to the chosen temperament.

The determination of this values is based on the following factors:

1. The length of time which the major third is sounded for.
2. The register of the notes that compounds the interval.
3. The rest of notes sounding at the same time.
4. The moment which both notes of the interval are playing at.

The tuning quality of the interval is most noticeable if it sounds for a reasonable length of time, if the notes belongs to the same register, especially in the middle or lower-middle register, and if there are no more notes sounding at the same time. On the contrary, tuning quality of the interval is less easily perceived if it is brief, if one of the notes is played earlier and has partly died away, if each note belong to a different register (or if both notes belong to the high register) and if other notes sound at the same time, particularly with dissonances.

The results of the evaluation for thirds occurred in all 24 preludes in major keys are summarised in the following table where each figure represents the number of times that each third formed on the respective note and considered with the respective prominence has been found in total for all preludes:

	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#
A	0	0	2	9	2	0	0	0	1	0	0	0
B	3	5	6	4	1	2	3	1	2	1	1	4
C	14	22	20	11	15	10	12	12	6	7	8	18
D	32	35	44	29	18	20	37	18	13	14	21	22
E	26	54	30	25	15	22	28	23	16	17	21	20
Method I	75	116	102	78	51	54	80	64	38	39	51	64
Method II	72	99	124	139	68	48	73	46	41	32	41	74

Two methods have been used for the evaluation. The first method is the most obvious and it simply consists to add together the five figures regardless of prominence. The advantage of this method lies in depending only on an objective analysis of the musical score according to certain simple rules. The weakness, however, is that it gives weight to events of no practical importance, that is to say, the occurrences of major thirds of category E whose tuning accuracy cannot be perceived.

The second method takes into account that thirds of category D are relatively numerous and their use shows only a weak correlation with their position in the circle of fifths. The more prominent thirds are relatively less numerous and show a stronger correlation. The available information is therefore likely to be more effectively used if categories A and B are counted as being of considerably greater relative importance than they are in Method I. Method II therefore counts thirds of category D once, C twice, B four times and A eight times. Thirds of category E are not taken into account.

The following table shows the deviations (expressed in cents) on respect Equal Temperament for the major thirds on each note of the circle of fifths in Werckmeister III temperament:

Werckmeister III	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#
Fifths	0	0	0	6	6	6	0	0	6	0	0	0
Major thirds	16	10	4	4	10	10	16	16	16	22	22	22

If this table is compared with the results of the evaluation, it is easy to observe a fairly strong correlation between both rows of figures, specifically for Method II. This means that more consonant thirds of the temperament (that is, those with a less deviation respect just intonation) are more prominent and more often occurred throughout each of the evaluated pieces of the work. These results suggest that, in the *Well-Tempered Clavier*, Bach used more freely those major thirds which are better tuned in Werckmeister III temperament and less freely those whose tuning is worse.

This correlation can be confirmed using two juxtaposed graphs:

1) A first graph where the horizontal axis contains the notes of the circle of fifths and the verical axis represents the frequency and prominence of major thirds according the Method 2, that is, the figures obtained in the previous table.
2) A second graph where teh horixontal axis also contains the notes of the circle of fifths and the vertical axis represents the values for the major third errors in Werckmesiter III temperament.

The basic similarity of the two graphs verifies the the strong correlation observed in the previous tables.

Moreover, another graph can be plotted combining the data shown in botrh previous graphs, that is, a third graph where the horizontal axis represents the error of major thirds in Werckmeister III temperament and the vertical axis represents the frequency and prominence of major thirds. Each degree of the scale now has a position on the graph depending on the two quantities.

The notes placed near the upper left part of the graph correspond to those with the smallest major third errors and also occurring most frequently. Similarly, the notes placed near the lower right part of the graph correspond to those with the biggest third errors and also occurring less fequently.

A linear regression is then calculated for these results and once the obtained line is plotted on the graph, a strong similarity can be observed between the points representing the degrees of the scale and the straight line, that is to say, the notes lie at points near the line.[57] This fact also corroborates the previous results.

The same graph is also plotted for Kelletat and Kellner temperaments where the degrees of the scale are further from the stright line than with Werckmeister temperament.

The layout for Werckmeister III temperament is shown in the following figure:

[57] The explanation of the details for the calculation of the linear regression is beyond my scope in this paper but it can be found in any book about statistics. An improvement of this method has been used by Claudio di Veroli in later works by him. *Vid.* Section "Claudio di Veroli (1980) [Tartini - Vallotti]".

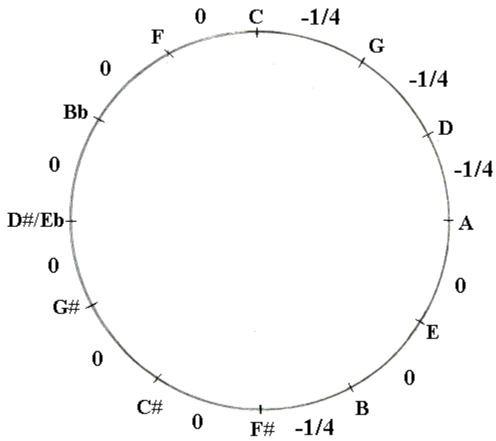

Figure 11 - Werckmeister III temperament

According to Bradley Lehman,[58] running his experimental data on 40 other temperaments and similarly plotting the results, Barnes's experiment strongly favors any temperaments that begin with F-C-G-D-A evenly tempered, as opposed to beginning with C-G-D-A-E.

John Barnes (1979, 2) [Barnes]

Apart from the results giving in the previous paragraph, there is no evidence that Werckmeister III was the temperament used by Bach. Really, Bach's temperament may have differed significantly from it.

Following his study in relation to Werckmeister III temperament, John Barnes proposed a modification of this temperament which can improve the effect of the dissonance in the same pieces of the *Well-Tempered Clavier*. He intended to reduce the level of the global dissonance more than any temperament did at that time.

Some apparent deviations in the use of certain intervals in the *Well-Tempered Clavier*, suggest that a more suitable temperament can be disovered. The results of the previous paragraph shows that the major thirds on G and D are used less than would be expected from their errors in Werckmeister III. Moreover, the major third on Ab is used more often than those on C# and F#. Taking into account its error in Werckmeister III, there is an apparent over-use of this interval.

The frequency and relative freedom of use of the Ab-major third suggest that, in Bach's temperament, this interval may have been better tuned that the other two on C# and F#. A small modification of Werckmeister III improves the Ab-major third and, at the same time, removes some of the distinctive purity of the F-major triad which, in Werckmeister III, unfortunately tends, when sounded, to make the ear more critical of the tuning errors of the other triads. The following table of deviations (also expressed in cents) of the intervals of fifth and major third on all the notes of the circle of fifths, defines this new temperament, which can be identified as Barnes temperament:

Barnes	Eb	Bb	F	C	G	D	A	E	B	F#	C#	G#
Fifths	0	0	4	4	4	4	4	0	4	0	0	0
Major thirds	14	10	6	6	10	10	14	18	18	22	22	18

Barnes temperament agrees with Werckmesiter III in having a group of tempered fifths centred on G and another tempered ith on B separated by a pure fifth.

[58] *Vid.* LEHMAN, Bradley: "Other 'Bach' temperaments"…, *op. cit.*

This pattern has the effect that the errors of the major thirds increase more gradually in the sharp direction than they do in the flat direction. It also has six tempered fifths compared with Werckmeister's four and each fifth is tempered by 1/6 of a Pythagorean comma, or approximately four cents. Barnes temperament may be set by ear almost as easily as Werckmeister III. Its layout is given in the following figure:

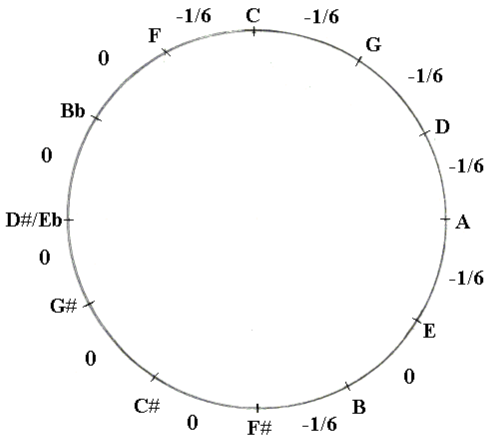

Figure 12 - Barnes temperament

The same graph obtained in the previous section[59] is plotted for Barnes temperament and a major linearity of the degrees of the scale can be noticed there, specifically for the note G#.

Claudio di Veroli (1980)[60] [Tartini - Vallotti]

Claudio di Veroli, as an answer to Barnes's article,[61] suggested the use of the popular Vallotti[62] temperament which is identical with Barnes temperament on 11 of the 12 notes, but with a slightly lower B. Really, Di Veroli is referring to Tartini-Vallotti temperament,[63] whose layout is given in the following figure:

[59] *Vid.* "John Barnes (1979, 1) [Werckmeister III]", (third graph).
[60] DI VEROLI, Claudio: "Bach's Temperament [correspondence with H.A. Kellner]". In: *Early Music*, Vol. 8, No. 1 (January 1980). Correspondence, p. 129. More information in: *Claudio Di Veroli: Harpsichord, organ, piano, fortepiano* [online]. In: <http://harps.braybaroque.ie/>. [June 2010].
[61] *Vid.* BARNES, John: "Bach's keyboard temperament...", *op. cit.*; and section "John Barnes (1979, 1) [Werckmeister III]".
[62] VALLOTTI, Francescoantonio: *Della Scienza Teoretica e Prattica della Moderna Musica.* Book 1: Padua: Giovanni Manfré, 1779. 167 pages. Books 2-4: MS, *I-Pca* [preliminary drafts of Books 1–4, *Pca*]; Modern edition: ZANON, G. & RIZZI, B. (eds.): *Trattato della moderna musica.* Padua, 1950.
[63] *Vid.* Section "Appendix 4: Description of some historical 'good' temperaments".

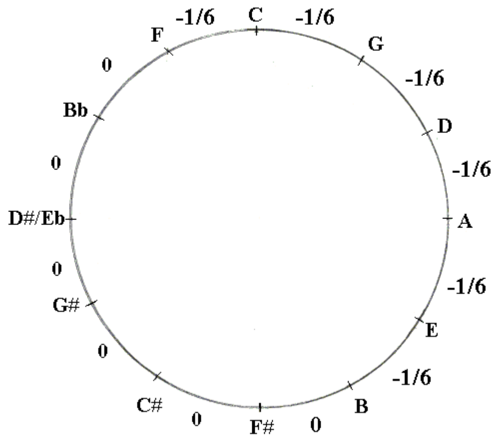

Figure 13 – Tartini - Vallotti temperament

Di Veroli adds that Vallotti temperament is "certainly better" than temperaments by Kelletat, Kellner, Werckmeister and other Baroque temperaments.

Herbert Anton Kellner (1981)[64]

Herbert Anton Kellner also answered to Di Veroli's letter since he does believe that his proposal is "a very good and appropriate tuning, distinctly better than any by Kirnberger and more suave than Werckmeister's" and, in his personal judgment, "it is the unique, natural and universal temperament for all 24 keys." Kellner substantiated his assertions by an "elaborate theoretical derivation"[65] and made reference to one of his previous works,[66] where he developed "the practical aspects and further reading concerning Bach's keyboard temperament."

Claudio di Veroli (1981)[67]

Claudio di Veroli also answered to Kellner's letter. Kellner's article is based on numerology (the study of coincidences among integer numbers) and Di Veroli doesn't agree that this system is a valid scientific method, although Bach used it in his compositions.

Di Veroli supports Barnes's article "where it is scientifically proved that Dr. Kellner's 'Bach' temperament is not the best suited for Bach's *Well-Tempered Clavier*." Barnes's or Vallotti's temperaments are more suitable than Werckmeister's or Kellner's

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64 KELLNER, Herbert Anton: "Bach's temperament [correspondence]". In: *Early Music*, Vol. 9, No. 1, Plucked-String Issue 1 (January 1981), p. 141.

65 DEHNHARD, W. & RITTER, G. (ed.): "Das ungleichstufige, wohltemperierte Tonsystem". In: *Bach-stunden. Festschrift für Helmut Walcha zum 70. Geburtstag überreicht von seinen Schülern.* Frankfurt am Main: Evangelischer Presseverband Hesse-Nassau, 1978, pp. 75-91. ISBN: 3-88352-014-4.

66 KELLNER, Herbert Anton: *Wie stimme ich selbst mein Cembalo?* Series: *Das Musikinstrument,* Vol. 19, Frankfurt am Main, 1976, 52 pages. 2nd edition: 1979, 61 pages. 3rd edition: Frankfurt am Mainz: Bochinsky, 1986, 73 pages. English translation: *The Tuning of my Harpsichord*. Series: *Das Musikinstrument,* Vol. 18, Frankfurt am Main: Verlag Das Musikinstrument, 1980, 54 pages. ISBN 3-920-112-78-4. Reprint of the English edition: Westport, CT (USA): Bold Strummer Ltd, December 1986. ISBN 3920112784, 54 pages. Japanese translation by Sumi Gunji: *Cembalo Choritsu: Bach no Hibiki o saigen suru*, Tokyo: Tokyo Ongakusha, 1990, 68 pages.

67 DI VEROLI, Claudio: "Bach's Temperament [observations]". In: *Early Music*, Vol. 9, No. 2 (April 1981), pp. 219-221. More information in *Claudio Di Veroli: Harpsichord, organ, piano, fortepiano* [online]. In: <http://harps.braybaroque.ie/>. [June 2010].

temperaments. Di Veroli proved these assertions following Barnes's criterion, based on the use of the dissonant intervals less frequently than the consonanrt ones.

Ralph Leavis (1981)[68]

Ralph Leavis also answerd to Barne's article, questioning the use of "his unequal temperament". Leavis "would ask whether he has tried it on the Chromatic Fantasia BWV 903 or the G minor organ Fantasia BWV 542/i." And also on "Bach's arrangement of Pergolesi's *Stabat mater*, with the voices and violins in F minor, the organ in D minor, and the harpsichord and string basses in E minor."

Peter F. Williams (1983)[69]

Peter F. Williams see the *Well-Tempered Clavier* "not only as a repertory of musical devices but as a summary of the law, a laying down of what is right", that is, an exposition of musical laws. "And this law is no a textbook abstraction but a kind of truth given to the talented by the Creator of all things, who expected the musician to use his talent to its full potential."

Regarding temperament, Williams asks himself which is the necessity of composing pieces in remote keys if there are no instruments able to play in such keys. He also asks himself if the pieces of the *Well-Tempered Clavier* where written originally in that keys. He really thinks that some of the pieces of the work where transposed before being included there. Thus, they can't sound well unless the clavier is individually tuned before playing each piece.

Some other interesting quotes of the article are reproduced bellow:

> "If the 48' was written as a set of instruction pieces for the young musician learning about temperament (this is the old chesnut about the *Well-Tempered Clavier*) then some crucial things are not made clear to him: chiefly, what exactly does 'well-tempered' mean (recently interpested evidence suggests that it meant 'equal temperament' after all), how is he to tune by it, and what precisely do the pieces demonstrate about the 24 keys other than how to get ypur fingers around them?
>
> Some of these 'secrets' could no doubt have been imparted by word of mouth amongst the Bach circle. But why in any case should a player need to play in all 24 keys? Nowhere else was he likely to find albums requiring him to play in G sharp minor and treating it as a key of equal status with, say, G minor. Certainly a few German organ tutors of the early 18th century sometimes gave the young organist chorales to play in organs able to play in such keys, but there cannot have been many church organs able to play them, i.e. on which they would have been tolerable. And we should not forget that Bach's extant music itself uses only some of the 24 keys.
>
> Anybody who sees in the '48' examples of how an intelligent composer uses keys that can never sound really well unless individually tuned should remember that some of the pieces in the '48' were most probably composed in other keys and transposed for these albums. So it is no use looking at the F sharp major pieces top see how Bach caters for the over-sharp third unless it can be demonstrated from the sources either that this was the key in which he originally composed the piece or that if he did transpose it to this key he had to alter the original in significant ways."

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[68] LEAVIS, Ralph: "Bach [correspondence]". In: *Early Music*, Vol. 9, No. 2 (April 1981), pp. 283-284.

[69] WILLIAMS, Peter F.: "J. S. Bach's *Well-Tempered Clavier*: A new approach – 1". In: *Early Music*, Vol. 11, No. 1, Tenth Anniversary Issue (January 1983), pp. 46-52.

Mark Lindley (1985)[70]

Mark Lindley thinks that some of the proposed reconstructions for Bach's temperament might have been acceptable to him. The problem is that "Bach never endorsed any mathematical tuning scheme and there is a fair amount of circumstancial evidence that he had none in mind, but rather tuned according to pragmatic criteria with leeway to accommodate such exigencies as the timbre of the instrument ar hand."[71] According to Lorenz Christoph Mizler[72], Lindley assumes that "in tuning harpsichords he knew how to temper so exactly and correctly that all keys sounded handsome and agreeable". Likewise, Carl Philipp Emanuel Bach "did not recommend tuning all the fifths uniformly."

Lindley consistently found that *The Well-Tempered Clavier* sounds better in an unequal temperament than in Equal Temperament. Even if Bach perhaps may not have preferred an unequal temperament, or may not have had a strong preference, he certainly accommodated and took advantage of it in this work; the different keys are in fact treated differently (notwithstanding that a few of the pieces were originally composed in different keys). Some fragments of the work are given as an examples of contrasts and differences between extreme keys, which can "probably dismiss the argument that *The Well-Tempered Clavier* must be performed with an equal-tempered keyboard. Lindley also gives more exemples of Bach's keyboard compositions which also proof this suggestion.

Mark Lindley (1993)[73]

Mark Lindley have developed an algebraic theory for tunings and temperaments. Starting from this theory as well as the study of the 18th-century temperaments, he has defined a model for a temperament which could be considered as Bach's presumed preferences.

According to the given definition for a 'good' temperament,[74] the deviation of the fifths can be expressed as follows:

$$\Delta q(n) = -\alpha(n)\xi_0$$

where $\alpha(n) \in \mathbf{Q}, 1 \le n \le n_0$. Likewise, the deviation for thirds respect to the just interval is:

$$\Delta t(n) = \varphi_0 - \xi_0 \sum_{i=n}^{n+3} \alpha(i)$$

[70] LINDLEY, Mark: "Bachs Stimmung des Klaviers". In: *Alte Musik als ästhetische Gegenwart: Bach, Händel, Schütz*. Bericht über den Internationalen Musikwissenschaftlichen Kongress, Stuttgart 1985. Gesellschaft für Musikforschung; herausgegeben von Dietrich Berke und Dorothee Hanemann. Kassel: Bärenreiter, 1987, Vol. 1, pp. 409-421. ISBN 3761807678. Reprint: "Klavierstemmingen van J.S. Bach". In: *Het Orgel*, Vol. 81, No. 6 (1985), pp. 409-421. English version: "J. S. Bach's Tunings". In: *The Musical Times*, Vol. 126, No. 1714 (December 1985), pp. 721-726.

[71] There is a more detailed analysis in LINDLEY, Mark: "Stimmung und Temperatur". In: ZAMINER, Frieder (ed.): *Hören, messen und rechnen in der frühen Neuzeit.* Series: Geschichte der Musiktheorie, Vol. 6. Darmstadt: Wissenschaftliche Buchgesellschaft, 1987, pp. 109-331.

[72] MIZLER, Lorenz Christoph: *Musikalischer Bibliothek*, Leipzig. Vol. 1/1-6 (1736-39), 64, 78, 49, 89, 77, 101pp.; Vol. 2/1-4 (1740-43), 158pp, pp.161-295, 176pp, 124pp.; Vol. 3/1-4 (1746-52), 778p.; Vol. 4/1 (1754), 182p.

[73] LINDLEY, Mark (with R. Turner-Smith): *Mathematical Models of Musical Scales: A New Approach.* Bonn: Verlag für systematische Musikwissenschaft GmbH, 1993. ISBN 3-922626-66-1.

[74] *Vid.* "Appendix 2: Definition of 'good' temperaments", section "'Good' temperaments based on Pythagorean intonation".

Lindley defines a JSB-system as a temperament of 12 notes based a on Pythagorean intonation in which the following conditions are fulfilled:

1) $-\frac{\xi_0}{6} \leq \Delta q(n) \leq 0$

$$-\frac{\xi_0}{6} \leq -\alpha(n)\xi_0 \leq 0$$

$$\frac{1}{6} \geq \alpha(n) \geq 0$$

2) $\Delta t(n) < \varphi_0$

$$\varphi_0 - \xi_0 \sum_{i=n}^{n+3} \alpha(i) < \varphi_0$$

$$\xi_0 \sum_{i=n}^{n+3} \alpha(i) > 0$$

This difference is “palpable”. A palpable difference is one of at least some small amount which can be set provisionally at 1 milioctave (1 thousandth of an octave).

3) Taking into account that $t(n) = T + \Delta t(n)$ where $T = \log \frac{5}{4}$

$$t(\pm n - n_0) \leq t(\pm(n+1) - n_0), n < 6$$

This means a gradual variation of the major thirds.

4) Similarly, $s(n - n_0) \leq s(n+1-n_0), n < 6$ and

$$s(-n - n_0 + 1) \leq t(-(n+1) - n_0 + 1), n < 5$$

This means a gradual variation of the semitones.

According to the last two conditions, thirds and semitones of the circle of fifths vary as indicated in the following figure:[75]

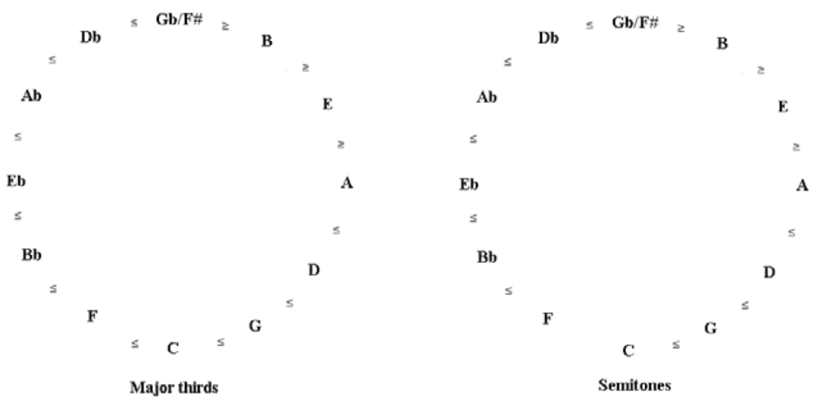

Figure 14 – Lindley conditions for thirds and semitones for a JSB-system

Two observations regarding JSB-systems are:

1) Observation 1: $\frac{\varphi_0}{4} < \Delta t(n)$

Proof: Starting from the first condition:

$$-\frac{\xi_0}{6} \leq \Delta q(n) \leq 0$$

and taking the value of $\Delta t(n)$ in function of $\Delta q(n)$:

[75] LINDLEY, Mark: *Mathematical Models of Musical Scales..., op .cit.*, p. 60.

$$\Delta t(n) = \varphi_0 - \xi_0 \sum_{i=n}^{n+3} \alpha(i) = \varphi_0 - \sum_{i=n}^{n+3} \Delta q(i)$$

the expression of the first condition can be rewriting as follows:

$$\varphi_0 - \frac{2}{3}\xi_0 \le \varphi_0 - \sum_{i=n}^{n+3} \Delta q(i) \le \varphi_0$$

Moreover, taking into account that

$$\xi_0 = \frac{12}{11}\varphi_0$$

the following result can be obtained, demonstrating that the observation is carried out:

$$\varphi_0 - \frac{2}{3}\xi_0 = \varphi_0 - \frac{2}{3}\frac{12}{11}\varphi_0 = \frac{3}{11}\varphi_0 > \frac{\varphi_0}{4}$$

$$\Delta t(n) > \frac{\varphi_0}{4}$$

2) Observation 2: Let's be $\Delta_{JSB}(n)$ and $\Delta_{ET}(n)$ the set of intervals of a JSB-system and an Equal Temperament respectively. Thus, for every JSB-system, the following condition is carried out:

$\Delta_{JSB}(n) - \Delta_{ET}(n) < 5 miliotaves, \forall n \in \{1,...,n_0\}$

Starting from the definition of both temperaments:

$$\Delta(n) = n \log\frac{3}{2} - \xi_0 \sum_{i=1}^{n} \alpha(i) + m(n)\log 2$$

$$\Delta(n) = n\left(\log\frac{3}{2} - \frac{\xi_0}{12}\right) + m(n)\log 2$$

The last difference can be calculated as follows:

$$\Delta_{JSB}(n) - \Delta_{ET}(n) = n\log\frac{3}{2} - \xi_0 \sum_{i=1}^{n} \alpha(i) - n\left(\log\frac{3}{2} - \frac{\xi_0}{12}\right) + m(n)\log 2 = \frac{\xi_0}{12}n - \xi_0 \sum_{i=1}^{n} \alpha(i)$$

Before demonstrating the last observation, two lemmas are set out and demonstrated bellow:

1) Lemma 1:

$$t(n) = T + \Delta t(n) = \log\frac{5}{4} + \Delta t(n) = \sum_{i=n}^{n+4} q(i) = 4Q + \sum_{i=n}^{n+4} \Delta q(i) = 4\log\frac{3}{2} + \xi_0 \sum_{i=n}^{n+4} \alpha(i) + m(n)\log 2$$

$$\Delta t(n) = 4\log\frac{3}{2} - \log\frac{5}{4} + m(n)\log 2 + \xi_0 \sum_{i=n}^{n+4} \alpha(i) = \log\frac{81}{80} + \xi_0 \sum_{i=n}^{n+4} \alpha(i)$$

$$\Delta t(n) = \varphi_0 + \xi_0 \sum_{i=n}^{n+4} \alpha(i)$$

2) Lemma 2: Starting from the third condition and applying the first lemma, a second lemma can be established. As an example, the first third C-E can be considered. According to the third condition, the third C-E is minor than G-B, that is to say:

$t(1) \le t(2)$

$T + \Delta t(1) \le T + \Delta t(2)$

Applying the second lemma:

$$\varphi_0 + \xi_0 \sum_{i=1}^{4} \alpha(i) \le \varphi_0 + \xi_0 \sum_{i=2}^{5} \alpha(i)$$

$\alpha(1) \le \alpha(5)$

Proceeding in a similar way for the rest of intervals of the circle, the following relations can be obtained:

$\alpha(1) \le \alpha(5) \le \alpha(9)$

$\alpha(2) \le \alpha(6) \le \alpha(10)$

$\alpha(3) \le \alpha(11) \le \alpha(7)$

$\alpha(4) \le \alpha(12) \le \alpha(8)$

Proof of the second observation: To show that

$\Delta_{JSB}(n) - \Delta_{ET}(n) < 5 milioctaves, \forall n \in \{1,\dots,n_0\}$,

Lindley constructs a system where

$$\alpha(n) = \begin{cases} -1/6, 1 \le n \le 6 \\ 0, 7 \le n \le 12 \end{cases}$$

and works out the differences for all the intervals of the circle with respect to the Equal Temperament. The results show that the largest distance between both systems is 1/2 of a Pythagorean comma, that is, 9.78 milioctaves. Nevertheless, if this system is "shifted" over by 1/4 of a Pythagorean comma, the results show that the maximal distance is 1/4 of a Pythagorean comma, that is 4.89 milioctaves. This amount is minor than 5 milioctaves, fulfilling the second observation.

The following table shows the results for the previous calculations:

Interval	ET	S_1	d(ET-S_1)(ξ_0)	S_2	d(ET-S_2) (ξ_0)	α_{ET}(n) (ξ_0)	α_{S1}(n) (ξ_0)	α_{S2}(n) (ξ_0)
C	0	0	0	1/4ξ_0	-1/6	1/12	1/6	1/6
G	log3/2-1/12ξ_0	log3/2-1/6ξ_0	1/12	log3/2-1/12ξ_0	-1/12	1/12	1/6	1/6
D	2log3/2-1/6ξ_0	2log3/2-1/3ξ_0	1/6	2log3/2-1/12ξ_0	0	1/12	1/6	1/6
A	3log3/2-1/4ξ_0	3log3/2-1/2ξ_0	1/4	3log3/2-1/4ξ_0	1/12	1/12	1/6	1/6
E	4log3/2-1/3ξ_0	4log3/2-2/3ξ_0	1/3	4log3/2-5/12ξ_0	1/6	1/12	1/6	1/6
B	5log3/2-5/12ξ_0	5log3/2-5/6ξ_0	5/12	5log3/2-7/12ξ_0	1/4	1/12	1/6	1/6
F#	6log3/2-1/2ξ_0	6log3/2-ξ_0	1/2	6log3/2-3/4ξ_0	1/6	1/12	0	0
C#	7log3/2-7/12ξ_0	7log3/2-ξ_0	5/12	7log3/2-3/4ξ_0	1/12	1/12	0	0
G#	8log3/2-2/3ξ_0	8log3/2-ξ_0	1/3	8log3/2-3/4ξ_0	0	1/12	0	0
Eb	9log3/2-3/4ξ_0	9log3/2-ξ_0	1/4	9log3/2-3/4ξ_0	-1/12	1/12	0	0
Bb	10log3/2-5/6ξ_0	10log3/2-ξ_0	1/6	10log3/2-3/4ξ_0	-1/6	1/12	0	0
F	11log3/2-11/12ξ_0	11log3/2-ξ_0	1/12	11log3/2-3/4ξ_0	-1/4	1/12	0	0

Systems which carry out the second observation, apart from the rest of lemmas and conditions, are JSB-systems whereas those which do not carry out it, are so-called "semi-JSB-systems".

According to the 18th-century German theoretical discussions of good temperaments, Lindley says that: "An irregular JSB-system with *n* palpably different sizes of semitones is better than one with *n-1* such different sizes. Given *n* such different sizes, then the less tempering for F#-A# is, the better the system is."

Finally, Lindley defines a theoretical Bach temperament as a JSB-system in which $\alpha(i) \in \{0,-1/6,-1/12\}$.

Some examples of theoretical Bach temperaments are certain irregular systems of Neidhardt and Sorge. Tartini-Vallotti temperament is like an irregular theoretical Bach temperament except that $\Delta t(n) \leq \varphi_0$ instead of $\Delta t(n) < \varphi_0$.

Mark Lindley (1994)[76] *[Lindley I & II]*

Mark Lindley's article is really a printed and expanded version of his presentation in the Michaelstein conference in 1994 which Mark Lindley took part in.

According to Carl Philipp Emanuel Bach, it is known that no one except Johann Sebastian Bach himself could tune the harpsichord to his satisfaction.[77] Starting with this premise, Lindley tries to find a temperament which could approximate to Bach's ideal style of tuning.

With regard to Bach's preferred style of unequal temperament and according to the contemporary writings, Mark Lindley is inclined to believe that:

1. There was some inequality among its semitones.
2. It was similar to the late- 17th- or early- 18th-century alternatives, the difference was really a matter of subtlety.

The works by authors such Andreas Werckmeister, Johann Georg Neidhardt or Johann David Heinichen justify the first conclusion whereas the second is more based on some remarks by Lorenz Christoph Mizler and Bach's son-in-law, Johann Christoph Altnickol. Nevertheless, the musical evidence is indispensable for both conclusions since the non-musical evidence is merely indicative but not conclusive.

Lindley proposes that the elaborate theoretical tuning models expounded by the likes of Neidhardt and Sorge were unable to capture the subtle nuances that Bach customarily achieved in tuning. Hi suggests, moreover, that the cause of the theoretical deficiency was that Sorge and Neidthardt would never split their basic unit of measurement for tempering, that is 1/12 of a Pythagorean comma. He thinks that this unit is too large to yield a theoretical model satisfying the following three conditions simultaneously:

1. A gradual variation of the major and minor thirds (or sixths).
2. The smallest major third C-E beats larger than pure.
3. The most heavily tempered thirds are impure by less than a *syntonic* comma and, consequently, Pythagorean thirds are excluded.

The second condition is based on the writings by all the German experts of Bach's day's. Likewise, Kirnberger also admitted that it was the Bach's own teaching.

The third condition was stated explicitly by Friedrich Wilhelm Marpurg and is implicit in all of Neidhardt's and Sorge's theoretical models of "good" unequal temperaments.

Lindley thinks that Bach's temperament achieves all these goals and a finer division of the Pythagorean comma is needed for this.

Moreover, in his theoretical model, Lindley designated the interval Db-F as the most heavily tempered major third. This statement is based on the following principles:

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[76] LINDLEY, Mark: "A Quest for Bach's Ideal Style of Organ Temperament". In: M. Lustig (ed.): *Stimmungen in 17. und 18. Jahrhunderts: Vielfalt oder Konfusion?* Michaelstein conference 1994. Michaelstein: Stiftung Kloster Michaelstein, 1997, pp. 45-67. Slovakian translation "Hladanie Bachovho idealneho stylu ladenia organu", *Slovenská Hudba,* Vol. 21, No. 3 (1997).

[77] BACH, Carl Philipp Emanuel: Letter to Forkel. Hamburg, 1774..., *op. cit.*

1. On average, this interval is tempered more than F#-A# in a large set of pertinent 18th-century models of unequal temperament.
2. The successive sharps and flats were tuned successively so low in the contemporary French style of unequal temperament. Moreover, it is known that Bach was strongly influenced by French music.

It is also clear for him that, in an unequal temperament, the fifths should sound as similar as possible whereas the thirds and semitones should vary in an expressive and gradual way.

The temperament proposed by Lindley is based in these conditions and premises. In the Michaelstein conference, a digital-tape recording was heard with some excerpts from Bach's organ music performed in the subtly unequal temperament designed by himself.

The examples on this tape were chosen from a larger pool of examples which showed that Bach's tendency to compose in many different keys, which everyone knows he did systematically in *The Well-Tempered Clavier*, was maintained to a considerable extent in his organ music. The musical demonstrations were unanimously accepted at the Symposium. Thus the tape showed that the proposed style of temperament works successfully and beautifully in Bach's organ music in every key that he used and it is unthinkable that he had to retune the organ for one key to another.

The proposed temperament constitutes a compromise amongst some late- 17th- and 18th-century theoretical models which Lindley believes were intended to represent practically the same style of tuning. His main premise is that Bach's concept of unequal temperament was a matter of qualities and not numbers.

In order to design the exact structure of the temperament, more additional premises were established. They were arrived at by considering the pertinent documentary evidence and by trying with several samples of Bach's organ and harpsichord music in various temperaments.

- The fifths C-G-D-A-E are tempered smaller than pure by the same amount *t*.
- The third C-E is tempered larger than pure by the least amount *a* and the third Db-F is tempered by the most amount *z*. This gives an asymmetrical model of the circle.
- The other major thirds among the naturals, G-B and F-A, are tempered uniformly by more than *a* according to the formula *a+i*, where *i* indicates an "increment".
- The thirds D-F#, A-C#, E-G# and B-D# are tempered each by an equal additional amount *s*, that is, *a+i+s*, *a+i+2s*, *a+i+3s* and *a+i+4s*.
- The thirds Bb-D, Eb-G and Ab-C are also tempered each by a different additional amount *f*. That is, *a+i+f*, *a+i+2f* and *a+i+3f*. The difference between s and f is given by the asymmetrical aspect of the circle and it is expected *f* to be larger than *s*.
- The thirds Ab-C and F#-A# are tempered less than Db-F in a uniformly way, that is, according to the formula *z-j*.
- The difference in tempering between B-D# and F#-A# is *j* rather than *s*. This premise is imposed by a mathematical consideration. It can be demonstrated that the amount *s* yields an Equal Temperament. Thus, the following equation can be assigned to the tempering of B-D#:

$$z - 2j = a + i + 4s$$

The scheme obtained is as follows:

C#	G#	D#	A#	
a+i+2s	a+i+3s	z-2j= a+i+4s	z-j	
A	E	B	F#	(C#)
a+i	a	a+i	a+i+s	a+i+2s
F	C	G	D	(A)
z	z-j= a+i+3f	a+i+2f	a+i+f	a+i
Db	Ab	Eb	Bb	(F)

Finally, the other fifths of the circle would have to be tempered less than *t* since the other major thirds would all be tempered more than C-E, that is, larger than *a*. The exact amount for these intervals can be calculated according to the method explained for temperaments defined by intervals of thirds, that is, solving a system of linear equations.[78] Thus, the following mathematical procedure is developed in order to yield a theoretical model for Bach's temperament.

Let $x_i, 1 \leq i \leq 12$ be the amounts of tempering for the 12 fifths o the circle. Thus, for each third, the following equations can be defined:

$$C-E: \quad a = 11 - x_1 - x_2 - x_3 - x_4$$
$$G-B: \quad a+i = 11 - x_2 - x_3 - x_4 - x_5$$
$$D-F\#: \quad a+i+s = 11 - x_3 - x_4 - x_5 - x_6$$
$$A-C\#: \quad a+i+2s = 11 - x_4 - x_5 - x_6 - x_7$$
$$E-G\#: \quad a+i+3s = 11 - x_5 - x_6 - x_7 - x_8$$
$$B-D\#: \quad a+i+4s = 11 - x_6 - x_7 - x_8 - x_9$$
$$F\#\text{–}A\#: \quad z-j = 11 - x_7 - x_8 - x_9 - x_{10}$$
$$Db-F: \quad z = 11 - x_8 - x_9 - x_{10} - x_{11}$$
$$Ab-C: \quad a+i+3f = 11 - x_9 - x_{10} - x_{11} - x_{12}$$
$$Eb-G: \quad a+i+2f = 11 - x_{10} - x_{11} - x_{12} - x_1$$
$$Bb-D: \quad a+i+f = 11 - x_{11} - x_{12} - x_1 - x_2$$
$$F-A: \quad a+i = 11 - x_{12} - x_1 - x_2 - x_3$$

Taking into account that $x_1 = x_2 = x_3 = x_4 = t$, the first half of previous system can be solved in the following way:

$$C-E: \quad a = 11 - 4t$$
$$G-B: \quad a+i = 11 - 3t - x_5 \Rightarrow x_5 = t - i$$
$$D-F\#: \quad a+i+s = 11 - 2t - (t-i) - x_6 \Rightarrow x_6 = t - s$$
$$A-C\#: \quad a+i+2s = 11 - t - (t-i) - (t-s) - x_7 \Rightarrow x_7 = t - s$$
$$E-G\#: \quad a+i+3s = 11 - (t-i) - 2(t-s) - x_8 \Rightarrow x_8 = t - s$$
$$B-D\#: \quad a+i+4s = 11 - 3(t-s) - x_9 \Rightarrow x_9 = t - i - s$$

The following equation

$$F\#\text{–}A\#: \quad z - j = 11 - 2(t-s) - (t-i-s) - x_{10}$$

can be solved taking into account that $z - 2j = a + i + 4s$,

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[78] *Vid.* Section “Appendix 6: Definition of temperaments according to the deviation of thirds”.

yielding $x_{10} = t - s - j$.

For the rest of the system, it can be proceeded in the following way:

$$F - A: \quad a + i = 11 - x_{12} - 3t \Rightarrow x_{12} = t - i$$

$$Bb - D: \quad a + i + f = 11 - x_{11} - (t - i) - 2t \Rightarrow x_{11} = t - f$$

$$Eb - G: \quad a + i + 2f = 11 - x_{10} - (t - f) - (t - i) - t \Rightarrow x_{10} = t - f$$

$$Ab - C: \quad a + i + 3f = 11 - x_9 - 2(t - f) - (t - i) \Rightarrow x_9 = t - f$$

Finally, the last equation

$$Db - F: \quad z = 11 - x_8 - 3(t - f)$$

can be solved taking into account that $z - j = a + i + 3f$

and yielding $x_8 = t - i - j$.

C#	t-s	G#	t-i-s	D#	t-s-j	A#		
a+i+2s		a+i+3s		z-2j= a+i+4s		z-j		
A	t	E	t-i	B	t-s	F#	t-s	(C#)
a+i		a		a+i		a+i+s		a+i+2s
F	t-i	C	t	G	t	D	t	(A)
z		z-j= a+i+3f		a+i+2f		a+i+f		a+i
Db	t-i-j	Ab	t-f	Eb	t-f	Bb	t-f	(F)

It can be noticed that two results have been obtained for the values of x_8, x_9 and x_{10} and the following system can also be defined:

$$x_8 = t - s = t - i - j$$

$$x_9 = t - i - s = t - f$$

$$x_{10} = t - s - j = t - f$$

which is equivalent to

$$s = j + i$$

$$f = s + i$$

$$f = s + j$$

yielding the following mathematical relations:

$$s = 2i$$

$$f = 3i$$

$$j = i$$

This lets to write the mathematical expressions of the amounts of all the intervals only as a functions of t, a and i.

C#	t-2i	G#	t-3i	D#	t-3i	A#		
a+5i		a+7i		a+9i		a+10i		
A	t	E	t-i	B	t-2i	F#	t-2i	(C#)
a+i		a		a+i		a+3i		a+5i
F	t-i	C	t	G	t	D	t	(A)
a+11i		a+10i		a+7i		a+4i		a+i
Db	t-2i	Ab	t-3i	Eb	t-3i	Bb	t-3i	(F)

According to the properties of good temperaments, the following equivalence can be also taken into account:

$$a + 5i + a + 1 + a + 11i = 21$$
and i can be expressed as a functions of a:
$$i = \frac{21 - 3a}{17}$$
Starting form the equation
$$z - j = a + i + 3f$$
replacing j and f by their values as a function of i obtained above
$$z - i = a + i + 9i$$
z can be also rewritten as a function of i:
$$z = a + 11i$$
Finally, if
$$a = 7 - \frac{17}{3}i$$
$$z = 7 + \frac{16}{3}i$$

Likewise, the values o t can be obtained starting from the following properties also given for good temperaments:
$$a = 11 - 4t$$
which corresponds to the first equation of the linear system obtained above, from where t can be obtained as a function of i:
$$t = \frac{11 - a}{4}$$

Summing up, the following expressions have been obtained:
$$i = \frac{21 - 3a}{17}$$
$$z = 7 + \frac{16}{3}i$$
$$t = \frac{11 - a}{4}$$
as well as the following table of values for i, z and t, for each integer value arbitrarily attributed to a:

a	i	z	t
3	12/17=0.71	183/17=10.8	2
4	9/17=0.53	167/17=9.82	7/4=1.75
5	6/17=0.35	151/17=8.88	3/2=1.5
6	3/17=0.18	135/17=7.94	5/4=1.25
7	0	7	1

In theory, an infinite number of intermediate possible temperaments could be obtained according to the pattern given. As musicians, the following ranges for a and z can be considered:

- $3 \leq a \leq 4$
- $9 \leq z \leq 10.5$

Moreover, long experience confirmed that Bach's music in particular sounds better if C-E is tempered by 3 units or more than if it is tempered by less than 3 units.

Finally, if *z* takes the value 10.5, the following values are obtained for *a*, *i* and *t*:[79]

a= 3.3, i=0.652941176, t=1.925

and the following scheme is yielded for the temperament[80]:

C#	0.62	G#	-0.03	D#	-0.03	A#				
6.5		7.9		9.1		9.8				
A	1.93	E	1.27	B	0.62	F#	0.62	(C#)		
4.0		3.3		4.0		5.3		6.5		
F	1.27	C	1.93	G	1.93	D	1.93	(A)		
10.5		9.8		7.9		5.9		4.0		
Db	0.62	Ab	-0.03	Eb	-0.03	Bb	-0.03	(F)		

Figure 15 - Lindley I temperament

According to Lindley, this temperament has a reeadily perceptible degree of inequality and can be set quite quickly since this scheme havs practically pre fifths for the chain Ab-Eb-Bb-F.

After this, Lindley analyses 17 temperaments by seven authors yielding some improvements for them. As an example, Lindley gives an algebraically derived "Average Neidhardt" temperament which is an average of both Neidhardt 1732 - 3rd circle II - Small city / Neidhardt 1724 I - Village and Neidhardt 1732 – 5th circle VIII - Big city / Neidhardt 1724 II - Small city temperaments.[81] Its definition is given in the following figure:

```
        C#  0.5   G#  0.5  D#  0.5  A#  0.5  (E#)
       7.5        8.5      8.5       9
      A   1.5   E   0.5  B   1.0  F#  0.5  (C#)
     4.5        3.5       5        6
    F   0.5   C   2.0  G   2.0  D   2.0  (A)
   9          9        7.5      6
  Db  0.5   Ab  0.5  Eb  0.5  Bb  0.5  (F)
```

Figure 16 - Lindley II - Average Neidhardt temperament (*schismata*)

The previous layout is equivalent to the following circular scheme:

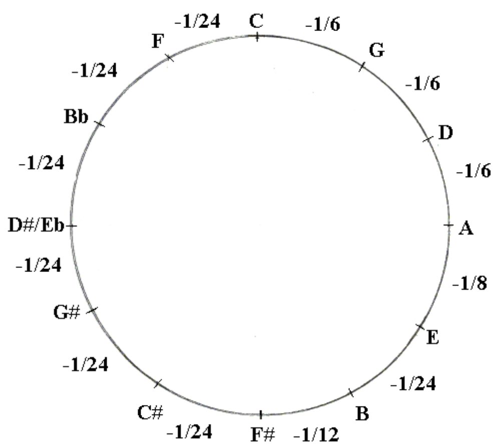

Figure 17 - Lindley II – Average Neidhardt temperament

[79] Approximations by Mark Lindley.

[80] Mark Lindley approximates –0.03 by 0.

[81] Vid. LINDLEY, Mark: "A Quest for Bach's Ideal Style of Organ Temperament"..., *op. cit.*, figure 16.

Andreas Sparschuh (1999)[82] [Sparschuh]

Andreas Sparschuch was the first person in modern times to suggest that the drawing at the top of the title page of Bach's *Well Tempered Clavier* means something in relation to a temperament. He interprets the scroll in this way:

- Each loop of the diagram is ascribed to one interval following the circle of fifths.
- Each kind of loop is interpreted in the following way: simple loops are interpreted as pure fifths whereas loops with squiggles are interpreted as fifths beating once per second (simple squiggles) or fifths with 0.5 beats per second (double squiggles).
- The diagram is read from left to right starting with the note A which corresponds to the extremes of the scroll. Thus, the first loop corresponds to E, and so forth.
- Finally the frequency of A is ascribed to 420 Hz.
- The rest of frequencies are determined following the circle of fifths taking into account the values of the beat rates.

Using the equations which relate the beat rates and the frequencies of the intervals,[83] an equation for each interval of the circle of fifths can be formulated. Some octave leaps downwards are also considered where appropriate. Thus a system of linear equations can be formulated in the following way:

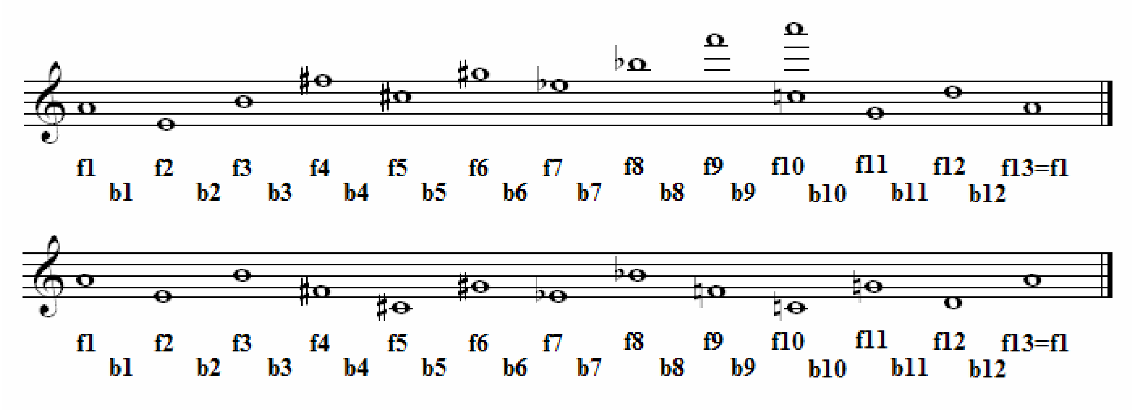

Figure 18 – Andreas Sparschuh's algorithm

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[82] SPARSCHUH, Andreas: "Stimm-Arithmetik des wohltemperierten Klaviers". In: *Jahrestagung der Deutsche Mathematiker-Vereinigung*, 1999, pp. 154-155.

[83] *Vid.* "Appendix 5: Definition of temperaments by beat rates".

$$\begin{cases} \frac{3}{4}f_1 - f_2 = b_1 \\ \frac{3}{2}f_2 - f_3 = b_2 \\ \frac{3}{2}f_3 - f_4 = b_3 \\ \frac{3}{4}f_4 - f_5 = b_4 \\ \frac{3}{2}f_5 - f_6 = b_5 \\ \frac{3}{4}f_6 - f_7 = b_6 \\ \frac{3}{2}f_7 - f_8 = b_7 \\ \frac{3}{2}f_8 - f_9 = b_8 \\ \frac{3}{2}f_9 - f_{10} = b_9 \\ \frac{3}{16}f_{10} - f_{11} = b_{10} \\ \frac{3}{2}f_{11} - f_{12} = b_{11} \\ \frac{3}{4}f_{12} - f_1 = b_{12} \end{cases}$$

In the previous linear system of equations, f_i is the frequency of each note of the circle and b_i, which is given for $1 \le i \le 12$, is the value of the beat rate for each interval. The frequencies match the notes as indicated in the first staff of Figure 18. Moreover, $f_{13} = f_1$ and b_{12} corresponds to the beat rate of the fifth ascribed to the end of the diagram whose value has to be determined. The complete values of b_i are:

$$b_i = \{1,1,1,0,0,0,0.5,0.5,0.5,0.5,0.5,b_{12}\}$$

One the one hand, equations related to descendent fourths have the form:

$$\frac{3}{4}f_i - f_{i+1} = \frac{b_i}{2}$$

On the other hand, equations related to ascendent fifths have the form:

$$\frac{3}{2}f_i - f_{i+1} = \frac{b_i}{2}$$

At first, equations can be rewritten so that all coeficients are integer, that is:

$$3f_i - 4f_{i+1} = 2b_i$$
$$3f_i - 2f_{i+1} = b_i$$

In order to work in the same octave and to generalize the procedure with the method given in "Appendix 5: Definition of temperaments by beat rates" (which is equivalent to that given by John Charles Francis[84]), some intervals should be inverted. According to the properties exposed in the same appendix, the following equivalences should be applied to the previous system of equations:

1) One descendent fourth can be converted in one ascendent fifth which beats at the same rate:

[84] *Vid.* Section "John Charles Francis (2005, 1) [Francis II & III]".

$$3f_i - 2f_{i+1} = 2b_i$$

2) One ascendent fifth can be converted in one descendent fourth which beats twice as faster:

$$3f_i - 4f_{i+1} = 2b_i$$

After this, the system of equations can be rewriting as follows:

$$\begin{cases} 3f_1 - 4f_2 = 4b_1 \\ 3f_2 - 2f_3 = 2b_2 \\ 3f_3 - 4f_4 = 2b_3 \\ 3f_4 - 4f_5 = 2b_4 \\ 3f_5 - 2f_6 = b_5 \\ 3f_6 - 4f_7 = 2b_6 \\ 3f_7 - 2f_8 = b_7 \\ 3f_8 - 4f_9 = b_8 \\ 3f_9 - 4f_{10} = b_9/2 \\ 3f_{10} - 2f_{11} = 2b_{10} \\ 3f_{11} - 4f_{12} = 2b_{11} \\ 3f_{12} - 2f_1 = 2b_{12} \end{cases}$$

At this moment, all equations of the system take the same form as the system given in the appendix, that is:

$$3f_i - 2a_i f_{i+1} = 2b_i$$

and it can be solved using the expression given there. Now the frequencies correspond to the notes given in the second staff of Figure 18.

Nevertheless, since f_1 is given, the system can be solved more easily:

$$f_{i+1} = \frac{3}{2a_i} f_i - \frac{b_i}{a_i}, 1 \le i \le N-1$$

Finally, the value of b_{12} can be worked out according to f_{12} and f_1:

$$b_{12} = \frac{3}{2} f_{12} - f_1$$

Starting with $f_1 = 250Hz$, the values of the frequencies of the rest of the notes are:

$$f_i = \{250, 374, 280, 420, 314, 470, 352, 264, 396, 297, 445, 333.5\}$$

$$b_{12} = 0$$

According to the principles of this method and after all values of beat rates have been calculated, the layout of Sparschuh's temperament (expressed as beat rates) can be represented in the following way:

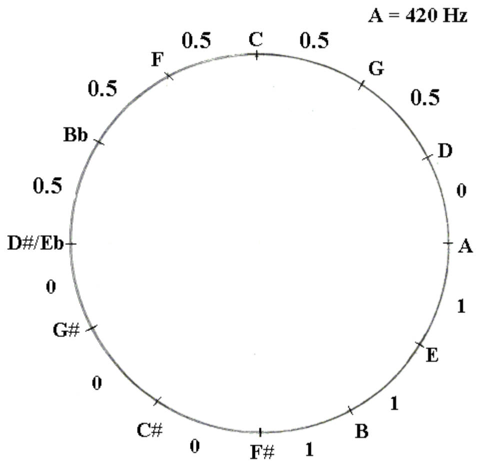

Figure 19 - Sparschuh temperament

Martin Jira (2000)[85] [Jira I & II]

Martin Jira presents two experimental temperaments[86] of his own devising which are identified by himself as “open”[87] and “closed”.[88] Both of them has the same tempering for all diatonic notes whereas they are variously tempered in notes with accidentals. The tempering of the diatonic notes consists of four fifths reduced by 1/6 of a Pythagorean comma between C and E, and two pure fifths at C-F and E-B.

On the one hand, the "open" temperament allows one fifth wider than pure which is placed between G# and Eb, the same place where the wolf fifth is placed in meantone temperaments. The size of the interval is the same as some of the intervals in Werckmeister’s temperaments, that is, 1/12 of a Pythagorean comma wider than pure. This temperament has a wider range of expressive variety among the keys, with greater variation of size among the thirds.

On the other hand, the "closed" temperament avoids intervals wider than a pure fifth.

Some writers absolutely refuse the existence of any fifth wider than pure[89] in temperaments which could be valid for Bach’s repertoire. Nevertheless, Jira allows the use of one of these kind of fifths in the “open” temperament.

The rest of the book analyses how these two strategies of tempering (“open” and “closed” intonation) interact with the complete Bach keyboard repertoire. In Jira’s opinion, some of Bach’s music works better in the “open” type of temperament and some of it in the “closed” type.

The layouts for both Jira's temperaments are shown in the following figures:

[85] JIRA, Martin*: Musikalische Temperaturen und Musikalischer Satz in der Klaviermusik von J. S. Bach* . Tutzing: Verlag Hans Schneider, 2000. 319 pages with CD. ISBN 3-795210-04-6.

[86] *Arbeitstemperaturen*.

[87] *Offene Arbeitstemperatur*.

[88] *Geschlossene Arbeitstemperatur*.

[89] *Äberschwebende Quinten*.

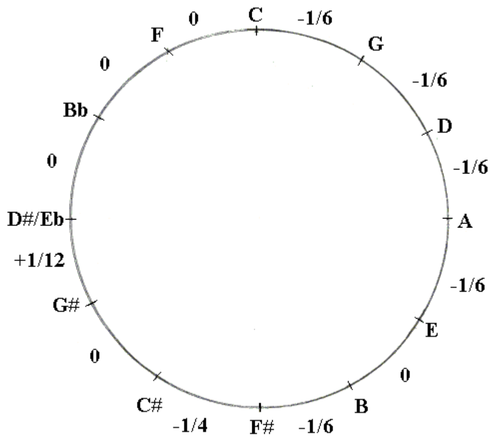

Figure 20 - Jira I temperament – "Open" temperament or *offene Arbeitstemperatur*.

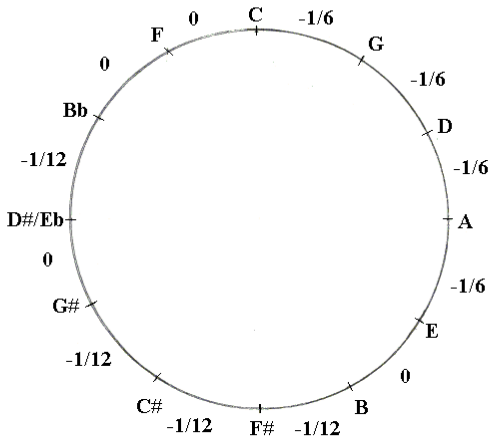

Figure 21 - Jira II temperament – "Closed" temperament or *geschlossene Arbeitstemperatur*.

Michael Zapf (2001)[90] [Zapf]

The clavichord specialist Michael Zapf met Sparschuh and reconsider the interpretation of Bach's diagram. Although Zapf was also convinced the diagram was indeed a tuning system, he was sceptical regarding Sparschuh's initial interpretation, considering it doesn't match the historical tuning practice. Zapf subsequently made a proposal of his own and also interpreted the loops of the scroll as beat rates. The principles of his method are very similar to those by Spasrschuh with some differences:

- As Sparschuh made, simple loops are interpreted as pure fifths whereas loops with squiggles are interpreted as fifths beating once per second (simple squiggles) or fifths with 0.5 beats per second (double squiggles).
- The diagram is also read from left to right but Zapfs starts with the note C instead of A.
- Both ends of Bach's diagram are included in the analysis in order to close the circle-of-fifths.
- Finally the frequency of C is ascribed to 250 Hz.

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[90] ZAPF, Michael: A reasonable revision of Sparschuch's work informally proposing in 2001. Several postings in: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. Several files posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>. [June 2010].

As was explained for Sparschuh tuning, using the equations which relate the beat rates and the frequencies of the intervals,[91] an equation for each interval of the circle of fifths can be formulated. Some octave leaps downwards are also considered where appropriate. Thus a system of linear equations can be formulated in the following way:

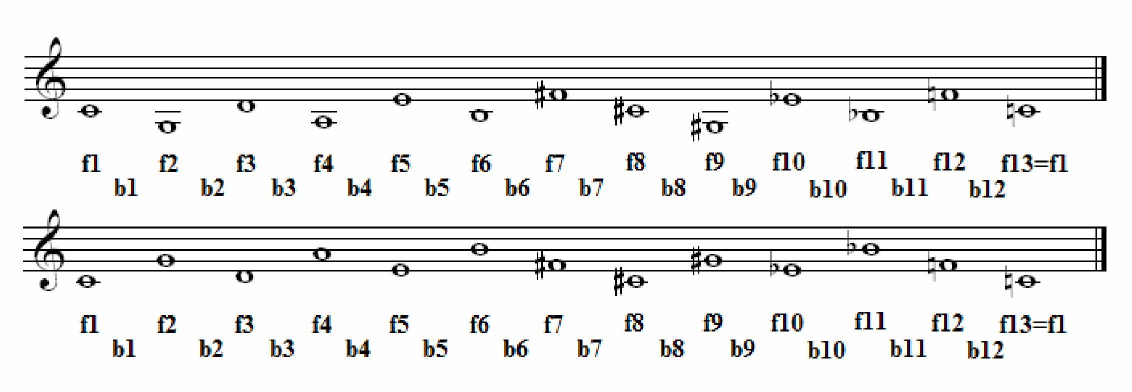

Figure 22 - Michael Zapf's algorithm

$$\begin{cases} \frac{3}{4}f_1 - f_2 = \frac{b_1}{2} \\ \frac{3}{2}f_2 - f_3 = \frac{b_2}{2} \\ \frac{3}{4}f_3 - f_4 = \frac{b_3}{2} \\ \frac{3}{2}f_4 - f_5 = \frac{b_4}{2} \\ \frac{3}{4}f_5 - f_6 = \frac{b_5}{2} \\ \frac{3}{2}f_6 - f_7 = \frac{b_6}{2} \\ \frac{3}{4}f_7 - f_8 = \frac{b_7}{2} \\ \frac{3}{4}f_8 - f_9 = \frac{b_8}{2} \\ \frac{3}{2}f_9 - f_{10} = \frac{b_9}{2} \\ \frac{3}{4}f_{10} - f_{11} = \frac{b_{10}}{2} \\ \frac{3}{2}f_{11} - f_{12} = \frac{b_{11}}{2} \\ \frac{3}{4}f_{12} - f_1 = \frac{b_{12}}{2} \end{cases}$$

In the previous linear system of equations, f_i is the frequency of each note of the circle and b_i, which is given for $1 \le i \le 12$, is the value of the beat rate for each interval. The frequencies match the notes as indicated in the first staff of Figure 22. Moreover, $f_{13} = f_1$ and b_{12} corresponds to the beat rate of the fifth ascribed to the end of the diagram whose value has to be determined. The complete values of b_i are:

$$b_i = \{1,1,1,0,0,0,0.5,0.5,0.5,0.5,0.5,b_{12}\}$$

One the one hand, equations related to descendent fourths have the form:

[91] *Vid.* “Appendix 5: Definition of temperaments by beat rates”.

$$\frac{3}{4}f_i - f_{i+1} = \frac{b_i}{2}$$

On the other hand, equations related to ascendent fifths have the form:

$$\frac{3}{2}f_i - f_{i+1} = \frac{b_i}{2}$$

At first, equations can be rewritten so that all coeficients are integer, that is:

$$3f_i - 4f_{i+1} = 2b_i$$

$$3f_i - 2f_{i+1} = b_i$$

In order to work in the same octave and to generalize the procedure with the method given in "Appendix 5: Definition of temperaments by beat rates" (which is equivalent to that given by John Charles Francis[92]), some intervals should be inverted as it has also made for Sparschuh's method.

Proceding similarly than in Sparschuh's method, the system of equations can be rewriting as follows:

$$\begin{cases} 3f_1 - 2f_2 = 2b_1 \\ 3f_2 - 4f_3 = 2b_2 \\ 3f_3 - 2f_4 = 2b_3 \\ 3f_4 - 4f_5 = 2b_4 \\ 3f_5 - 2f_6 = 2b_5 \\ 3f_6 - 4f_7 = 2b_6 \\ 3f_7 - 4f_8 = 2b_7 \\ 3f_8 - 2f_9 = 2b_8 \\ 3f_9 - 4f_{10} = 2b_9 \\ 3f_{10} - 2f_{11} = 2b_{10} \\ 3f_{11} - 4f_{12} = 2b_{11} \\ 3f_{12} - 4f_1 = 2b_{12} \end{cases}$$

At this moment, all equations of the system take the same form as the system given in the appendix, that is:

$$3f_i - 2a_i f_{i+1} = 2b_i$$

and it can be solved using the expression given there. Now the frequencies correspond to the notes given in the second staff of Figure 22.

Nevertheless, since f_1 is given, the system can be solved more easily:

$$f_{i+1} = \frac{3}{2a_i}f_i - \frac{b_i}{a_i}, 1 \le i \le N-1$$

Finally, the value of b_{12} can be worked out according to f_{12} and f_1:

$$b_{12} = \frac{3}{2}f_{12} - f_1$$

Starting with $f_1 = 250Hz$, the values of the frequencies of the rest of the notes are:

$$f_i = \begin{Bmatrix} 250, 374, 280, 419, 314.25, 471.376, 353.531, \\ 264.898, 396.848, 297.386, 445.578, 333.934 \end{Bmatrix}$$

$$b_{12} = 0.9$$

[92] *Vid.* Section "John Charles Francis (2005, 1) [Francis II & III]".

According to the principles of this method and after all values of beat rates have been calculated, the layout of Zapf's temperament (expressed as beat rates) can be represented in the following way:

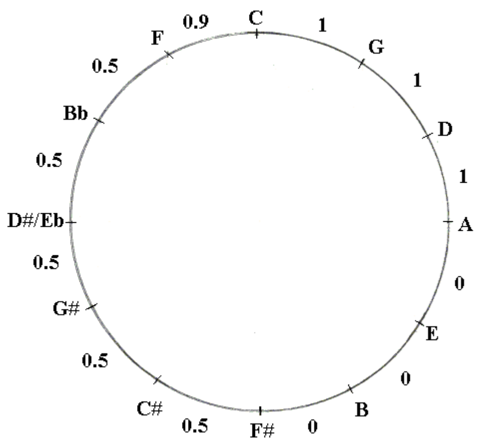

Figure 23 - Zapf temperament

There is another interpretation of Zapf's temperament where it is expressed as fractions of a Pythagorean comma. This approximation is given by Thomas Dent[93] and is shown in the following figure:[94]

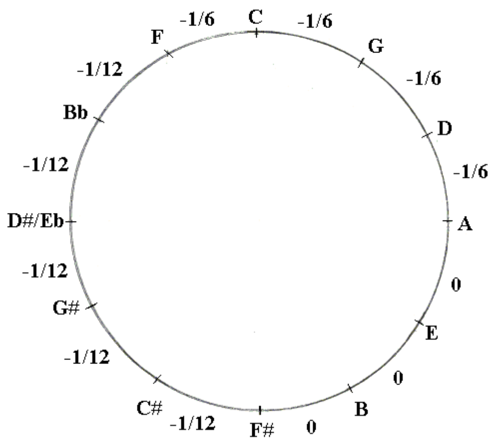

Figure 24 – Zapf temperament – Dent's approximation

Paul Simmonds (2003)[95]

According to Keith Briggs,[96] Paul Simmonds made reference to Michael Zapf's[97] interpretation of Bach's diagram.

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[93] DENT, Thomas: "Zapf, Lehman and other Clavier-Well-Temperaments". In: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. July 2005. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. File posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>, 7 pages. [June 2010].

[94] The mathematical procedure for this approximation is given in the "Appendix 5: Definition of temperaments by beat rates", section "Approximation by fractions of a comma".

[95] SIMMONDS, Paul: Letter to the Editor. In: *Early Music Review*, Vol. 88 (March 2003).

[96] *Vid.* Section "Keith Briggs (2003)".

[97] *Vid.* Section "Michael Zapf (2001) [Zapf]".

Keith Briggs (2003)[98] [Briggs]

Keith Briggs also makes reference to Michael Zapf's[99] research about Bach's diagram. Briggs checked the maths of Zapf's interpretation and he added that "it is possible to go further and deduce an absolute pitch for the starting note". Assuming that the final fifth F-C beats once per second, the pitch of C can be calculated according to the rest of beat rates and frequencies and must be about 127 Hz (126.8 Hz exactly). Thus this corresponds to about A=425 Hz.

The applied process is similar to Zapf's one but Briggs have applied the modifications added by Johann Charles Francis,[100] which are also set out in this paper,[101] according to them it is possible to determine all pitches of the scale. The difference is that Zapf's assumes an initial frequency for the the note working as the beginning of the process instead of a beat rate for the end of the diagram.

The value obtained by Briggs can be easily demonstrated using the mathematical process explained in the appendix.[102] Thus, as was explained for Sparschuh and Zapf tunings and using the equations which relate the beat rates and the frequencies of the intervals,[103] an equation for each interval of the circle of fifths can be formulated. Some octave leaps downwards are also considered where appropriate. Thus a system of linear equations can be formulated in the following way:

Figure 25 – Keith Briggs' algorithm

[98] BRIGGS, Keith: Letter to the Editor. In: *Early Music Review*, Vol. 90 (May 2003). Also available in: BRIGGS, Keith: "Letter from Keith Briggs to the editor, Early Music Review. Published May 2003". In: *Keith Briggs* [online]. May 2003. In: <http://keithbriggs.info/bach-wtc.html>. [December 2010].
[99] *Vid.* Section "Michael Zapf (2001) [Zapf]".
[100] *Vid.* "John Charles Francis (2005, 1) [Francis II & III]".
[101] *Vid.* "Appendix 5: Definition of temperaments by beat rates".
[102] *Vid.* "Appendix 5: Definition of temperaments by beat rates".
[103] *Vid.* "Appendix 5: Definition of temperaments by beat rates".

$$\begin{cases} \frac{3}{2}f_1 - f_2 = \frac{b_1}{2} \\ \frac{3}{4}f_2 - f_3 = \frac{b_2}{4} \\ \frac{3}{2}f_3 - f_4 = \frac{b_3}{2} \\ \frac{3}{4}f_4 - f_5 = \frac{b_4}{4} \\ \frac{3}{2}f_5 - f_6 = \frac{b_5}{2} \\ \frac{3}{4}f_6 - f_7 = \frac{b_6}{4} \\ \frac{3}{4}f_7 - f_8 = \frac{b_7}{4} \\ \frac{3}{2}f_8 - f_9 = \frac{b_8}{2} \\ \frac{3}{4}f_9 - f_{10} = \frac{b_9}{4} \\ \frac{3}{2}f_{10} - f_{11} = \frac{b_{10}}{2} \\ \frac{3}{4}f_{11} - f_{12} = \frac{b_{11}}{4} \\ \frac{3}{2}f_{12} - 2f_1 = \frac{b_{12}}{2} \end{cases}$$

In the previous linear system of equations, f_i is the frequency of each note of the circle and b_i, which is given for $1 \leq i \leq 12$, is the value of the beat rate for each interval. The frequencies match the notes as indicated in the first staff of Figure 25. Moreover, $f_{13} = 2f_1$ and b_{12} corresponds to the beat rate of the fifth ascribed to the end of the diagram. The complete values of b_i are:

$$b_i = \{1,1,1,0,0,0,0.5,0.5,0.5,0.5,0.5,1\}$$

One the one hand, equations related to descendent fourths have the form:

$$\frac{3}{4}f_i - f_{i+1} = \frac{b_i}{2}$$

On the other hand, equations related to ascendent fifths have the form:

$$\frac{3}{2}f_i - f_{i+1} = \frac{b_i}{2}$$

At first, equations can be rewritten so that all coeficients are integer, that is:

$$3f_i - 4f_{i+1} = 2b_i$$

$$3f_i - 2f_{i+1} = b_i$$

In order to work in the same octave and to generalize the procedure with the method given in "Appendix 5: Definition of temperaments by beat rates" (which is equivalent to that given by John Charles Francis[104]), some intervals should be inverted as it has also made for the methods by Sparschuh and Zapf.

Proceding similarly than in the methods by Sparschuh and Zapf, the system of equations can be rewriting as follows:

---

[104] *Vid.* Section "John Charles Francis (2005, 1) [Francis II & III]".

$$\begin{cases} 3f_1 - 2f_2 = b_1 \\ 3f_2 - 4f_3 = b_2 \\ 3f_3 - 2f_4 = b_3 \\ 3f_4 - 4f_5 = b_4 \\ 3f_5 - 2f_6 = b_5 \\ 3f_6 - 4f_7 = b_6 \\ 3f_7 - 4f_8 = b_7 \\ 3f_8 - 2f_9 = b_8 \\ 3f_9 - 4f_{10} = b_9 \\ 3f_{10} - 2f_{11} = b_{10} \\ 3f_{11} - 4f_{12} = b_{11} \\ 3f_{12} - 4f_1 = b_{12} \end{cases}$$

At this moment, all equations of the system take the same form as the system given in the appendix, that is:

$$3f_i - 2a_i f_{i+1} = b_i$$

and it can be solved using the expression given there. Now the frequencies correspond to the notes given in the second staff of Figure 25 and its obtained values are:

$$f_i = \begin{Bmatrix} 253.63, 379.94, 284.71, 426.56, 319.92, 479.88 \\ 359.91, 269.81, 404.46, 303.22, 454.58, 340.81 \end{Bmatrix}$$

According to the principles of this method and after all values of beat rates have been calculated, the layout of Briggs temperament (expressed as beat rates) can be represented in the following way:

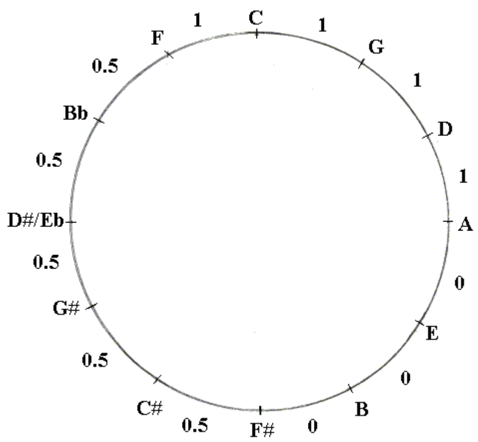

Figure 26 - Briggs temperament

Sergio Martínez (2003-2004)[105] [Kirnberger II]

In my previous work, I proposed that the Kirnberger II temperament was the most suitable temperament to be used in Bach's *Well-Tempered Clavier*. The temperament was choosen among several of the main good German temperaments of the 18th century. The conclusion is obtained according to a study of the sensory

[105] MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software…, op. cit.;* MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia...", *op. cit.*

dissonance which is also based on a mathematical model for the dissonance given by William A. Sethares.[106] This is the same criterion has been taken into account in the current research. Thus the principles of this article will be explained and developed in the following sections of this paper.[107] The definition of the Kirnberger II temperament is shown bellow:

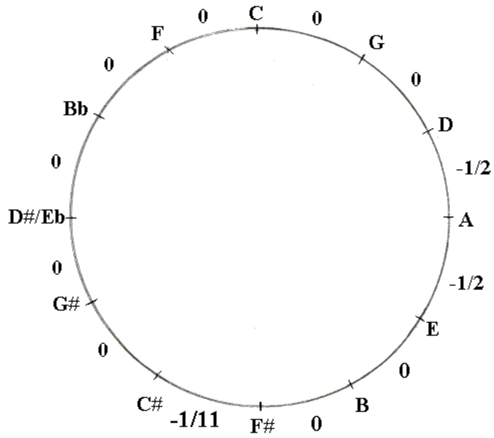

Figure 27 - Kirnberger II temperament

John Charles Francis (2004)[108] [Francis I]

John Charles Francis assumes that the first prelude of the *Klavierbüchlein für Wilhelm Friedemann Bach* by Johann Sebastian Bach (BWV 924),[109] which was prepared in January 1720, following the tenth birthday of his oldest son, was explicitly constructed to provide a pedagogic tuning aid for his young son, Wilhelm Friedemann. In fact the complete work was composed with didactic objectives.

On the other hand, Francis starts with two more assumptions:

1) Johann Nikolaus Forkel, writing in 1802, noted that J. S. Bach always tuned his own keyboard and that the procedure did not take him more than fifteen minutes.[110]

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[106] SETHARES, William A. *Tuning,…, op. cit.*

[107] *Vid.* section "Evaluation according to dissonance theory" and appendices.

[108] FRANCIS, John Charles: "The Keyboard Temperament of J. S. Bach". In: *Eunomios. An open online journal for theory, analysis and semiotics of music* [online]. 25 June 2004. In: <http://www.eunomios.org/contrib/francis1/francis1.html>. [June 2010]. Also available in: *Bach Cantantas Website* [online]. 19 November 2010. In: <http://www.bach-cantatas.com/Articles/Keyboard-Temperament[Francis].htm>. [June 2010]. Extensive supporting musical examples in MP3 format are also available in: *Bach Cantantas Website* [online]. In: <www.bach-cantatas.com/Articles/Keyboard-Temperament[Francis]-Music.htm>. [June 2010].

[109] BACH, Johann Sebastian: *Klavierbüchlein für Wilhelm Friedmann Bach.* Kassel: Bärenreiter-Verlag Karl Vötterle GmbH & Co. KG, 1962. Edited by Wolfgang Plath. Critical commentaries in: Neue Bach-Ausgabe, Klavier- und Lautenwerke Band 5, *Kritischer Bericht / Critical Commentary.* Kassel: Bärenreiter-Verlag Karl Vötterle GmbH & Co. KG, 1962. BA 5163. A facsimile edition is also available: BACH, Johann Sebastian: *Clavier-Büchlein for Wilhelm Friedmann Bach.* London: Oxford University Press, 1959; Re-edition: New York: Da Capo Press, 1979. With a preface by Ralph Kirkpatrick.

[110] FORKEL, Johann Nicolaus: *Über Johann Sebastián Bachs Lebens, Kunst und Kunstwerke.* Leipzig: Hoffmeister und Kühnel, 1802, x, 69 p., 2p mus exx. Reprints and re-editions: Leipzig: C. F. Peters, 1855, viii, 48, 4 p.; Josef Müller-Blattau (ed.): Augsburg: Bärenreiter, 1925. 112p. Kassel: Bärenreiter, 1950. 103 p.; Max F. Schneider (ed.): *Bücher der Weltliteratur*, Vol. 10. Basel: Haldimann, 1945. 151 p.; Hans R. Franzke (ed.): Hamburg: Laatzen, 1950, 70p.; Frankfurt a. M.: H. L. Grahl 1950. x, 69p, 2p mus exx.; Ralph Kirkpatrick (ed.): *Über Johann Sebastian Bachs Leben, Kunst und Kunstwerke. (On Johann Sebastian Bach's Life, Genius, and Works).* New York: C. F. Peters Corp., 1950. x, 72p. [With commentary in English]; Walther Vetter(ed.): Berlin: Henschel, 1966. 156p. Kassel [u.a.]: Bärenreiter

2) Friedrich Wilhelm Marpurg, writing in 1776, related that J. S. Bach had confided his tuning method to his pupil Johann Philipp Kirnberger, who was expressly required to tune all the thirds sharp.[111]

After the analysis and interpretation of the bass line of the piece, Francis concludes thaht the harpsichord could be tuning according the following method:

1. Tune the fifths C-G-D-A-E-B justly.
2. Tune the interval D-F# justly.
3. Temper the E and B such the three intervals A-E, E- B and B-F# are equally good, that is, reduced by 1/3 of a *syntonic* comma.
4. Retune F#, such that the interval B-F# is a just fifth (the third D-F# is tempered by 1/3 of a *syntonic* comma wider) (optional) and continue tuning a just circle of fifths to D#/Eb.
5. Reflecting the upward pointing *doppelt cadence u. mordant* on G# in BWV 924, widen the fifth C#-G# by 1/3 of a *syntonic* comma (optional).
6. Tune the Bb and F as just fifths starting from D#/Eb. The remaning fifth F-C results to be tuned narrowing by slightly more than 1/3 of a syntonic comma, that is, it is exactly reduced by 1/3 of a *syntonic* comma and the *schisma* (1/3+1/11=14/33 of a *syntonic* comma).

The tuning of the fifths between C and B is given by the succession of the six notes which results of taking the notes marked with trills in order of occurrence in the bass line of the first three bars and preceded by the opening note (C).

The assumption of tuning just fifths is motivated by several factors: the facility of tuning in just fifths, the rapidity with which can be done and the historical precedent of employing just fifths. The velocity and facility of the method justifies one of the first hypothesis according to it, Johann Sebastian Bach was able to tune his harpsichord in no more than fifteen minutes.

The trills are of two different kinds: the first four are *mordants* and the latter is a so-called *doppelt cadence u. mordant*. The leading tones of the *mordants* are must be tuned from the notes already tuned. They are also ordered according to the circle of fifths: F#-C#-G#-E#. These notes are related by thirds with the previous notes. Since semitone relations cannot be tuned directly by ear with any useful accuracy, there is but one practical possibility: to tune by thirds. At first, the assumption will be to tune these thirds justly. The first third can be tuned from D resulting a wolf fifth between B and

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[Lizenz d. Henschelverlag], 1968, 1982, 162p.; *Felix*, Werner (ed.) *Johann Sebastian Bach. Erbe und Gegenwart,* pp. vii-xxxii, 1-288. Leipzig: Offizin Andersen Nexö, 1975. xxxii, 413p; Leipzig: Offizin Andersen Nexö, 1983. xxxii, 383p.; Axel Fischer (ed.): Kassel [u.a.]: Bärenreiter, 1999. 144p. ISBN: 3-7618-1472-0; Claudia Maria Knispel (ed.): Henschel, 2000. xiii, 184p. ISBN: 3-89487-352-3; Christoph Wolff, Michael Maul (eds.): *Bach-Dokumente. Supplement zu Johann Sebastian Bach. Neue Ausgabe sämtlicher Werke*, Vol. 7 (2008), x, 228p. Kassel [u.a.]: Bärenreiter, 2008. x, 228p. ISBN: 978-3-7618-1925-8; English translations: Stephenson? or Kollmann? (ed.): *Life of John Sebastian Bach; with a critical view of his compositions.* London: T. Boosey & Co., 1820, xi, 116p.; "Life of John Sebastian Bach; with a critical view of his compositions". In: *The Musical World. A weekly record of musical science, literature and intelligence.* Vol. 15, No. 15/250-252 (NS. 8/158-160), 254-257 (NS. 8/162-165) & 259-261 (NS. 167-169). London: Novello, 1841; "Life of John Sebastian Bach: with a Critical View of his Compositions". In: *Dwight's Journal of Music.A paper of art and literature,* Vol. 8, No. 4-16. Boston, Mass.: Balch, 1855-56; "Life of John Sebastian Bach". In: Th*e Musical World. A weekly record of musical science, literature and intelligence.* Vol. 43, No. 11, 14-16, 21, 24, 25, 28-30, 32-35 & 41. London: Novello, 1865; Charles Sanford Terry (ed.): *Johann Sebastian Bach, His Life, Art and Work.* London: Constable, 1920. xxxii, 321p; Reprint: New York: Da Capo. Press, 1970; Vienna House 1974, xxxii, 321p.

111 SCHULZE, Hans-Joachim *et al.* (ed.): *Bach-Dokumente*, Kassel & Leipzig: Bärenreiter, 1972. Reedition: *Bach-Dokumente,* In: *Die Neue Bach-Ausgabe*, Supplement. Kassel & Leipzig: Bärenreiter, 1982-2010, Vol. 3 No. 815.

F#. The intervals C-E and G-B are Pythagorean major thirds, intervals which are reserved for remote keys. The third step of the method of tuning described above solves this problem. The tuning of the notes E and B is given by the two notes placed at the end of the third and fifth measure and marked with the so-called *doppelt cadence u. mordant* that points downwards, that is, the direction in which the E and B must be tempered.

After this, the rest of thirds can be tuned following the circle of fifths, as it is explained in the fourth step. Nevertheless, the fifth B-F# can be retuned justly before tuning C#, G# and D#/Eb. This step is considered as optative by Francis.

By analogy, the *doppelt cadence u. mordant* which points upwards and placed in the the G# placed at the end of the fourth measure, can be interpreted as a widening of the interval E-G#. Consequently, the fifth G#-D#/Eb is narrowed by the same amount. This step is also considered to be optative by Francis.

The tuning of the remaining fifths can be done in different ways:

1) Tune Bb justly from Eb and then tune F justly from Bb; this is a valid option.
2) Tune Bb justly from Eb and tune F justly from C; this option can be discounted since it gives rise to an inappropriate Pythagorean third on the interval Bb-D.
3) Tune F justly from C and then tune Bb justly from F; this option can also be discounted as it gives rise to two inappropriately Pythagorean thirds on the intervals F-A and Bb-D.
4) Tune F and Bb such that the intervals Eb - Bb, Bb – F and F - C are equal; this option compromises the just fifths unnecessarily and leads to an undulating pattern of widening thirds with two peaks. Accordingly, it can be discounted.

The first option is the only valid solution and, consequently, it is chosen by Francis for the method of tuning.

Taking into account that two of the steps are considered to be optative, four temperaments are yielded by Francis in his article but, finally, the best solution is the temperament obtained taken into account the both optional steps.

On the one hand, the tempering corresponding to the upward pointing *doppelt cadence u. mordant* yields a temperament in which all their tetrachords are tuned in a different way, that is, all they are unique. Otherwise, several tetrachords result to be identical. A more explicit key colour and variety is obtained for temperaments in which all their tetrachords are different.

On the other hand, the tempering which considers the retuning of F# such that the interval B-F# is a just fifth, yields a temperament without just major thirds. This is in concordance with the second assumpttion of the article: Friedrich Wilhelm Marpurg related that J. S. Bach had confided his tuning method to Kirnberger, who was expressly required to tune all the thirds sharp.

In all cases, Francis places the *schisma* in the fifth F-C. Thus, if this fifth is also narrowed by 1/3 of a *syntonic* comma, it results to be narrowed by “slightly more than 1/3 os a *syntonic* comma”. This means exactly that this fifth is narrowed by 1/3 of a *syntonic* comma and also by the *schisma*, that is to say, narrowed by 14/33 of a *syntonic* comma (1/3 + 1/11 = 14/33).

The layout of the resulting temperament is shown bellow:

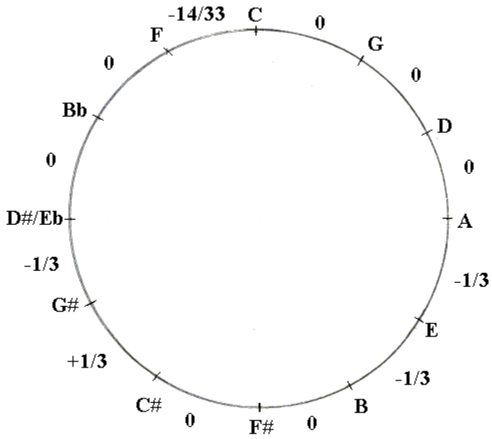

Figure 28 - Francis I temperament

Bradley Lehman (2005, 1)[112] [Lehman I & II]

Bradley Lehman believes that Johann Sebastian Bach notated a specific method of keyboard tuning but "he did not express it in our normally-expected formats of theory, or numbers, but rather drew a diagram for a practical hands-on sequence." He also believes that "its particular sound, as an integrated part of musical practice, has profound implications for all of Bach's instrumental and vocal music that uses keyboards: either with written-out parts or in the *basso continuo*."

Lehman refers to the decorative scroll placed at the top of the title page of the autograph manuscript of the first volume of *The Well-Tempered Clavier*, which is called "Rosetta Stone" by him, and he asserts that this graphic can be interpreted as a method of tuning. Nevertheless, he omits that other researchers had already made reference to the interpretation of this diagram as a method of tuning some years before him: Andreas Sparschuh (1999), Michael Zapf (2001), Paul Simmonds (2003) and Keith Briggs (2003).[113]

His own first preferred reading of the diagram (spring 2004) is based on the *syntonic* comma interpretation and it has an arbitrarily pure fifth at Bb-F. This temperament can be identified as Lehman I temperament and his layout is shown bellow:

[112] LEHMAN, Bradley. "Bach's extraordinary temperament: our Rosetta Stone – 1". In: *Early Music*, Vol. 33, No. 2 (February 2005), pp. 3-24. Also available in: *Johann Sebastian Bach's tuning* [online]. April 2006. In: <http://www-personal.umich.edu/~bpl/larips/outline.html#rosetta>. [June 2010]; LEHMAN, Bradley. "Bach's extraordinary temperament: our Rosetta Stone – 2". In: *Early Music*, Vol. 33, No. 5 (May 2005), pp. 211-231. Also available in: *Johann Sebastian Bach's tuning* [online]. April 2006. In: <http://www-personal.umich.edu/~bpl/larips/outline.html#rosetta>. [June 2010]. More information in: LEHMAN, Bradley: "Errata and clarifications for "Bach's extraordinary temperament: Our Rosetta Stone". Early Music, February and May 2005". In: *Johann Sebastian Bach's tuning* [online]. February 2007. In: <http://www-personal.umich.edu/~bpl/larips/errata.html> & <http://www.larips.com>. [January 2011].

[113] *Vid.* Sections "Andreas Sparschuh (1999) [Sparschuh]", "Michael Zapf (2001) [Zapf]", "Paul Simmonds (2003)" and "Keith Briggs (2003)". This omission is corrected and justified in LEHMAN, Bradley: "Errata and clarifications for "Bach's extraordinary temperament…", *op. cit.*

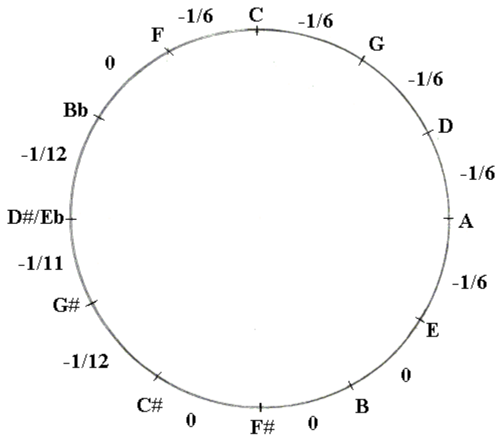

Figure 29 - Lehman I temperament

A later discussion with Ross Duffin and Debra Nagy conviced him to focus on the Pythagorean comma interpretation since it aligns better with the 18th century orchestral norm of the "55-division" of the octave. This new version of the temperament is discussed in several articles, in his Web site and in several postings on the Internet.[114]

At first, Lehman suggests that the diagram would have to be read downwards and after this, he suggests the following correspondence between the loops of the diagram and the intervals:

- The first three loops, which contains single spirals, match the chain A#-D#-G#-C# and these intervals are tempered as fifths reduced by 1/12 of a Pythagorean comma.
- The next three single loops, that is, without spiral, match the chain C#-F#-B-E and these intervals are tuned as pure fifths
- The remaining five loops, which contains elaborated squiggles, match the chain E-A-D-G-C-F and these intervals are tempered as fifths reduced by 1/6 of a Pythagorean comma. This chain of fifths gets both F-A and C-E thirds most nearly pure.
- Finally, the end of the diagram is associated to the fifth A#/Bb-F which is tempered as a fifth enlarged by the residual 1/12 of a Pythagorean comma. This interval is really a diminished sixth.

The tempering amount of 1/12 of a Pythagorean comma corresponds to one theoretical unit and the 1/6 of a Pythagorean comma to two units. Then it is very evident the correspondence between the loops and the units of impurity, that is, loops with double spiral corresponds to two units, loops with single spiral corresponds to one unit and single loops corresponds to zero units or pure intervals.

Moreover, in the last loop of the diagram (the first if it is tnverted), there is a handwritten letter 'C' linked to the capital 'C' of 'Clavier'. It can be a flourish of the title but, according to Lehman's interpretation of the Bach's diagram, it coincides with the position of the note 'C' in the fifth circle.

The resulting layout of this method of tuning is shown in the following graphic:

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114 *Vid.* Section "Bibliography" and the remaining sections that are dedicated to the same author. The complete information about his researches is included on his own Web site: LEHMAN, Bradley: *Johann Sebastian Bach's tuning* [online]. In: <http://www-personal.umich.edu/~bpl/larips/> & <http://www.larips.com>. [December 2010].

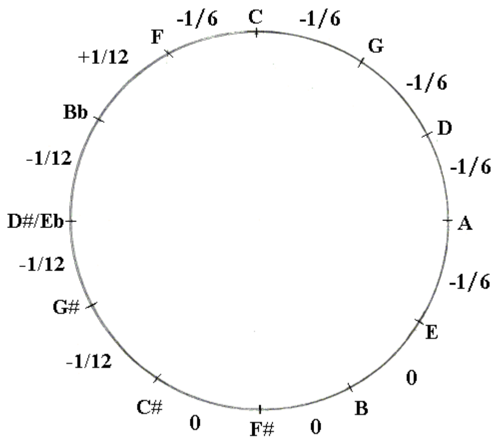

Figure 30 - Lehman II temperament

As well as the theoretical descrtiption of his temperament, Lehman sets out "a quick and practical bearing method" of tuning "by ear". Its instructions can be found in several publications on the Internet.[115]

Apart from this, in his Web site there exist a lot of practical information about his method of tuning and even more information about other "Bach" temperaments. For example, the following resources, among others, can be found there:

- Pages of recorded musical examples and 20-minute playlists of streaming audio, with performances by Bradley Lehman on harpsichords and pipe organs.
- Additional sample recordings are available variously on Last.fm, iLike or Facebook.
- More than a dozen recordings by other musicians using this "Bach/Lehman 1722" temperament: on harpsichords, fortepianos, pipe organs, digital organs, synthesizers, and more.
- A survey of the temperament's use in public performances and recordings by hundreds of musicians, and built into pipe organs and other instruments.
- There is a growing collection of video demonstrations, showing how to tune harpsichords by ear in this and several related temperaments.
- More historical information about tuning and temperaments, practical instructions, frequently asked questions (FAQ) and other sections with lot of documentation.

---

[115] LEHMAN, Bradley: "Practical temperament instructions by ear". In: *Johann Sebastian Bach's tuning* [online]. 29 March 2006. Posted in: <http://www-personal.umich.edu/~bpl/larips/practical.html>. [June 2010]. Other version: "A quick and practical bearing method, by ear". In: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. 29 June 2006. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. File posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>. [June 2010]. Other version: LEHMAN, Bradley: "The 'Bach temperament' and the clavichord". In: *Clavichord International*, Vol. 9, No. 2 (November 2005), pp. 41-46. Also available in: "The 'Bach temperament' and the clavichord". In: *Johann Sebastian Bach's tuning* [online]. October 2006. In: <http://www-personal.umich.edu/~bpl/larips/clavichord.html>. [June 2010].

John Charles Francis (2005, 1)[116] [Francis II & III]

John Charles Francis develops a systematic mathematical analysis of Bach's diagram. He interprets the loops of the graphic as the twelve intervals of the circle of fiths which can be associated in different ways. There are three different kinds of loops which are interpreted in terms of beats per second also giving three types of intervals:[117]

- The first three loops, which contains single spirals, correspond to intervals beating once per second.
- The next three single loops, that is, without spiral, correspond to pure intervals without beats.
- The remaining five loops, which contains elaborated squiggles, correspond to intervals beating twice per second.
- Finally, the end of the diagram can be associated to any of the previous types of intervals.

Starting with this principle, Francis establish the principles of the analysis according to several possibilities:

- Diagram orientation: left to right or right to left.
- Tuning direction: towards sharps or towards flats.
- Starting position: 12 possible positions.
- Beat rates for the interval closing the circle of fifths: 0, 1 or 2.

All these criteria gives a total of 144 possibilities to be analysed. The solution to the raised question would be to choose one or several practical temperaments which resulted from this analysis.

These 144 temperaments can be identified systematically depending on how they are derived from the scroll. If *x* is the value given for the end beat rate and the scroll is is reading starting at the *n*-th left loop, the systematic nomenclature can be established as a function of the values of *x* and *n* in the following way:

- Tuning the circle of fifths toward sharps and reading the scroll left to right (clockwise): Temperaments n-x, that is, 1-0 to 12-0, 1-1 to 12-1 and 1-2 to 12-2.
- Tuning the circle of fifths toward sharps and reading the scroll right to left (anticlockwise): Temperaments Rn-x, that is, R1-0 to R12-0, R1-1 to R12-1 and R1-2 to R12-2.
- Tuning the circle of fifths toward flats and reading the scroll left to right (clockwise): Temperaments Mn-x, that is, M1-0 to M12-0, M1-1 to M12-1 and M1-2 to M12-2.
- Tuning the circle of fifths toward flats and reading the scroll right to left (anticlockwise): Temperaments MRn-x, that is, MR1-0 to MR12-0, MR1-1 to MR12-1 and MR1-2 to MR12-2.

Each group of the previous scheme include 36 possibilities or temperaments. In total, 36*4=144 possibilities to be considered in the study.

Using the equations which relate the beat rates and the frequencies of the intervals[118], an equation for each interval of the circle of fifths can be formulated. Some

[116] FRANCIS, John Charles: "The Esoteric Keyboard Temperaments of J. S. Bach". In: *Eunomios. An open online journal for theory, analysis and semiotics of music* [online]. 9 February 2005. In: <http://www.eunomios.org/contrib/francis2/francis2.pdf>. [June 2010]. Also available in: *Keybard Tuning of Johann Sebastian Bach* [online]. 1 February 2005. In: <http://sites.google.com/site/bachtuning/literature>. [June 2010]. 49 pages.

[117] In the 18th century, beat rates could be determined easily using a pocket watch or pendulum clock.

[118] *Vid.* "Appendix 5: Definition of temperaments by beat rates".

octave leaps downwards are also considered where appropriate. Thus a system of linear equations for each temperament obtained can be formulated in the following way:

$$\begin{cases} 3f_1 - 2f_2 = b_1 \\ 3f_2 - 4f_3 = b_2 \\ 3f_3 - 2f_4 = b_3 \\ 3f_4 - 4f_5 = b_4 \\ 3f_5 - 2f_6 = b_5 \\ 3f_6 - 4f_7 = b_6 \\ 3f_7 - 4f_8 = b_7 \\ 3f_8 - 2f_9 = b_8 \\ 3f_9 - 4f_{10} = b_9 \\ 3f_{10} - 2f_{11} = b_{10} \\ 3f_{11} - 4f_{12} = b_{11} \\ 3f_{12} - 4f_1 = x \end{cases}$$

In the previous linear system of equations, f_i is the frequency of each note of the circle and b_i, which is given for $1 \le i \le 12$, is the value of the beat rate for each interval. The frequencies match the notes as indicated in Figure 31. Moreover, $f_{13} = f_1$ and b_{12} corresponds to the end beat rate, which can take three possible values: $x \in \{0,1,2\}$. This system of linear equations in twelve unknows has the same form as the system given in "Appendix 5: Definition of temperaments by beat rates" and it can be solved according to the method explained there.

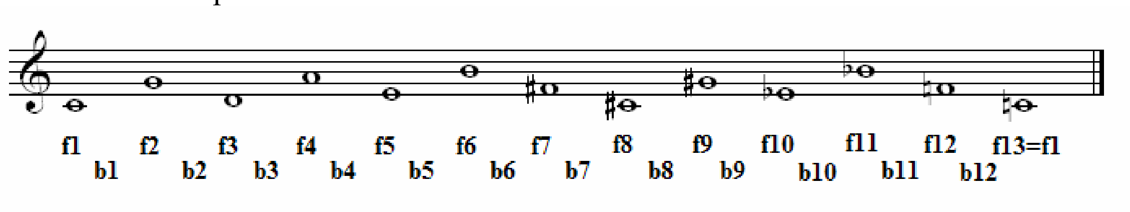

Figure 31 – John Charles Francis' algorithm

After solving these 144 systems of linear equations and evaluating the temperaments obtained, it is necessary to make a selection of the solutions which can be considered as appropriate for practical temperaments according to the tuning circumstances at Bach's time. According to the historical pitch standards,[119] the value of the frequency of A should be included within the range of 5 Hz of the mean *Cornet-ton* & *Cammerton* pitches. This value corresponds to the frequency f_9. Among all the temperaments obtained, this condition is only carried out in the following temperaments: 1-0, 6-0, 8-1, 9-1, 10-1, 3-2, 4-2, 7-2, 11-2, R1-0, R11-0, R2-1, R3-1, R8-1, R4-2 and R12-2.

Moreover, Francis realized that some temperaments are transpositions of others with a frequency difference of one semitone or one whole tone. The solutions which separated by a whole tone provides a means to tune keyboard instruments in *Cornet-ton* and *Cammerton* pitches, such that they can be used together with perfect intonation. Among the previous selected temperaments, only the following pairs carried out this condition:

- Temperament 9-1 (*Cammerton*) and its transposition 7-2 (*Cornet-ton*).

[119] *Vid.* "Appendix 7: Tuning circumstances at Bach's time".

- Temperament R2-1 (*Cammerton*) and its transposition R12-2 (*Cornet-ton*).

R12-2 was derived as a right to left reading of the diagram and it can be shown that mirror form of R2-1 corresponds to a left to right reading.

It is also tentative to think that the 'C' of 'Clavier' marked in the scroll shows the transpositions R2-1 (*Cammerton*) and R12-2 (*Cornet-ton*) are the Bach's preferred solution.

Noting that the major third on the tonic note is also part of the triad of its relative minor, it can be seen that the quality of the major third impacts both the quality of tonic major and its relative minor and, moreover, both share the same key signature. Thus the previous hypothesis can be confirmed by statistical analysis according to the frequency of occurrence of key signatures in Bach's clavier and organ works. The results of this analysis were correlated with the size of the thirds in each major key for the temperaments derived from the glyph and strong correlations were found for R12-2 and R2-1 temperaments. Likewise, the strongest correlation was obtained for R1-0 temperament, which corresponds to the mid-point of voth selcted temperaments.[120]

Finally, the layouts for both Francis' temperaments are shown in the following figures:

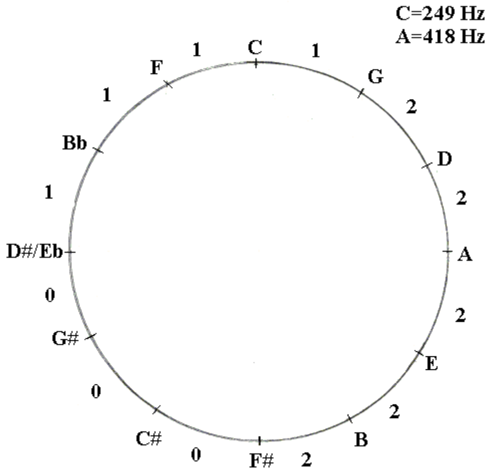

Figure 32 - Francis II - *Cammerton* temperament (R2-1)

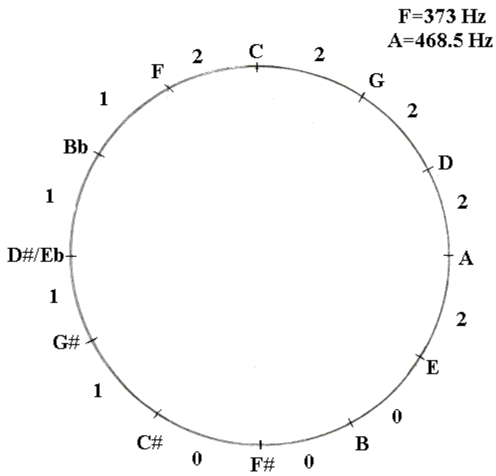

Figure 33 - Francis III - *Cornet-ton* temperament (R12-2)

The sequence of beat rates for each of the selected temperaments is:

[120] A more datailed explanation of this procedure is found in section "Temperaments and key signatures in Bach's works".

$b_i = \{1,2,2,2,2,2,0,0,0,1,1,1\}$, for *Cammerton* temperament (R2-1).

$b_i = \{2,2,2,2,0,0,0,1,1,1,1,2\}$, for *Cornet-ton* temperament (R12-2).

This *Cornet-ton* / *Cammerton* solution confirms a tentative hypothesis that the glyph represents two circles of fifths corresponding to the respective end points of the diagram. Both temperaments obtained are transposed by two places on the circle of fifths, that is a whole tone, and this requires adaptation to one of the beat rates. The difference in beat rates between *Cammerton* and *Cornet-ton* tuning procedures that occurs at the interval closing the circle of fifths is explicitily provided in the end of Bach's diagram. Thus the fifth corresponding to this point beats once per second for temperament R2-1 (*Cammerton*) and twice per second for temperament R12-2 (*Cornet-ton*).

Reading the diagram upwards, the circle of fifths for each of both solutions is:

R2-1: C-F-Bb-Eb/D#-G#-C#-F#-B-E-A-D-G

R12-2: Bb-Eb/D#-G#-C#-F#-B-E-A-D-G-C-F

For the tuning method in temperament R2-1, it can be considered that the starting point is C and the fifths can be tempered following the circle anticlockwise. On the contrary, for the tuning method in temperament R12-2, it can be considered that the starting point is F and the fifths can be tempered following the circle clockwise. In this sense, and according to the author, it can be also considered that one solution exists reading the glyph left to right while another exists reading the glyph right to left. With this solution, instruments at *Cornet-ton* pitch sound one tone higher than their *Cammerton* counterpants and, to coexist harmoniously in a ensemble, their temperament must be transposed downwards by a whole tone.

As a conclusion, the temperament for *The Well-Tempered Clavier* consists of two transposed *Cammerton* and *Cornet-ton* variants, indicated by respective left to right and right to left readings of Bach's diagram.

Francis' article also includes an analysis of the quality of the major and minor tetrachords, an interval analysis for thirds and fifths, a complete interval analysis and a comparison with historic temperaments in terms of Euclidian and correlation metrics.

John Charles Francis (2005, 2)[121]

John Charles Francis presented a second paper with the same results of the previous article but in a "more musician-friendly manner". The article also provides comparison with earlier interpretations proposed by Sparschuh, Zapf and Lehman, as well as more details in terms of comparison with other historical temperaments.

It is known that the best tuning practices at the time did not satisfy Bach. His son Carl Philipp Emanuel Bach noted that no one else could tune the harpsichord of his father to his satisfaction.[122]

[121] FRANCIS, John Charles: "Das Wohltemperirte Clavier. Pitch, Tuning and Temperament Design". In: *Eunomios. An open online journal for theory, analysis and semiotics of music* [online]. 10 July 2005. In: <http://www.eunomios.org/contrib/francis3/francis3.pdf>. [June 2010]. Also available in: *Bach Cantantas Website* [online]. 8 March 2006. In: <http://www.bach-cantatas.com/Articles/Das_Wohltemperirte_Clavier.htm>. [June 2010]. Also available in: *Keybard Tuning of Johann Sebastian Bach* [online]. 1 July 2005. In: <http://sites.google.com/site/bachtuning/literature>. [June 2010]. Also available in: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. 2 July 2005. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. File posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>. [June 2010]. 30 pages.

[122] BACH, Carl Philipp Emanuel: Letter to Forkel. Hamburg, 1774..., *op. cit.*

In this regard, Lindley proposes that the typical basic unit of measurement for tempering, that is 1/12 of a Pythagorean comma, is unable to satisfy the conditions which were defined by himself.[123] Thus in order to satisfy Lindley's conditions, Francis notes that Bach's temperament can achieve all these goals using a finer division of the Pythagorean comma.

Considering the sequence of beat rates of Francis temperaments:

$$b_i = \{0,0,0,1,1,1,1,2,2,2,2,2\}$$

it can be seen that the sum of the digits is 14.[124] According to the established approximation by fractions of a comma for temperaments defined by beat rates,[125] a theoretical tempering unit can be defined as 1/14th of a Pythagorean comma. Thus Francis' temperaments can be approximated by other ones with tempering fractions 1/7 and 1/14 (two and one units respectively). The resulting layouts for the approximations of temperaments R2-1 and R12-2 are shown in the following figures:[126]

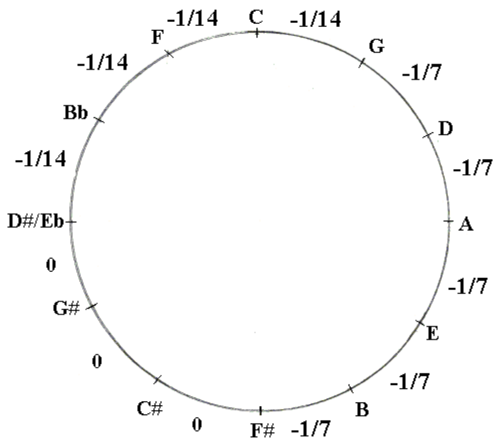

Figure 34 - Francis II - Cammerton temperament (R2-14P approximation)

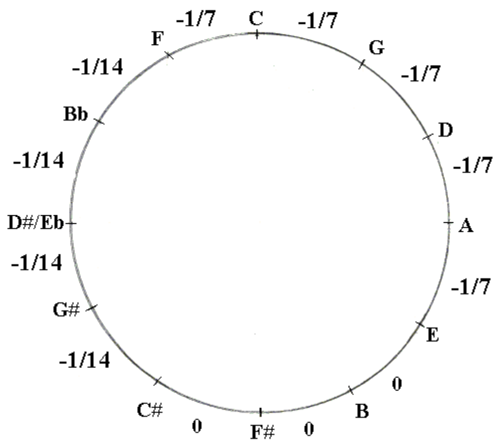

Figure 35 - Francis III - Cornet-ton temperament (R12-14P approximation)

The major third is given by a sequence of four consecutive fifths. Thus the width of the major thirds is determined by the beat rates of its four fifths, narrowing the

[123] LINDLEY, Mark: "A Quest for Bach's Ideal Style of Organ Temperament"..., *op. cit.,* pp. 45-67. *Vid.* Section "Mark Lindley (1994) [Lindley I & II]" for a more detailed explanation of Lindley's conditions.
[124] It can also be noted that, based on principles of gematria, "BACH" equates to 14 (2+1+3+8=14).
[125] *Vid.* "Appendix 5: Definition of temperaments by beat rates", section "Approximation by fractions of a comma".
[126] The nomenclature R2-14P and R12-14P is also defined by John Charles Francis in this article.

resulting interval. The following table shows the narrowing of the major thirds from the Pythagorean interval (expressed as 1/14 of a Pythagorean comma) which are obtained respectively for each group of four consecutive fifths troughout the circle of fifths:

Tempering of intervals on circle of fiths	Narrowing from Pythagorean third
0001	1
0011	2
0111	3
1111	4
1112	5
1122	6
1222	7
2222	8
2222	8

A progressive narrowing of the major thirds by one unit at a time is clearly shown. Thus there is a gradual and optimally smooth progression from worst to best thirds across the circle of fifths, so satisfying Lindley's first condition.

It can easily been noted that a sequence of four pure fiths does not occur. Thus the Pythagorean third is excluded, so satisfying Lindley's second condition. The worst case of the diagram, that is to say the widest third, consists of three consecutive pure fifths followed by a tempered interval.

Francis also proves that the smooth progression is also given for minor thirds and fifths. Likewise, significant distorsion occurs with other temperaments cretaed as approximations by other fractions of the Pythagorean comma, defeating the ideal of smooth, regular, transitions on the circle of fifths.

As a conclusion, it ids proved that temperaments based on the Bach's diagram interpreted as beat rates carry out Lindley's conditions and the 1/14 of a Pythagorean comma is the natural basic tempering unit to theoretically describe Bach's temperament. Larger fractions compromise the ideal of progressive gradual change in the size of thirds on the circle of fifths.

John Charles Francis (2005, 3)[127]

In this paper, John Charles Francis discusses the solution given by Bradley Lehman.[128] Lehman proposes that the glyph must be rotated to be correctly interpreted and this accords with a proposal of Francis. Likewise, Francis agrees that Bach indicated a "C" reference, as Lehman also noted.[129]

Moreover, Lehman's temperament corresponds to a modified 1/6-comma meantone solution with other fifths narrowed by 1/12-comma fractions. This is contradictory with Francis' theory, which defends a mathematical analysis based on beat rates (apart from the implication of both *Cammerton* and *Cornet-ton* solutions). However, Lehman's solution is closely correlated with Francis' temperaments in terms of beats per second. In Francis' opinion, Lehman's temperament "sounds decent" but it has several distorsions and imperfections. For example: there is a wide fifth Bb-F of 704

[127] FRANCIS, John Charles: "Review of article: 'Bach's extraordinary temperament: our Rosetta Stone – 1'". In: *Keybard Tuning of Johann Sebastian Bach* [online]. 14 February 2005. In: <http://sites.google.com/site/bachtuning/ReviewLehman.pdf>. [June 2010]. 7 pages.
[128] LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
[129] FRANCIS, John Charles: "The Esoteric Keyboard Temperaments…", *op. cit.*

cents whose two extra cents must result in further narrowing (detuning) of the other fifths.

Francis adds that Bach's pupil Johann Philipp Kirnberger constructed a temperament strongly correlated with the key occurrence of Bach's works and "Lehman's current solution exhibits a lower correlation" with it.[130]

Thomas Dent (2005)[131] [Dent I, II & III]

Thomas Dent intends to prove that opposing decisions made by different authors in interpreting Bach's diagram can lead to very similar solutions. Likewise substantially different tunings can result from the same diagram as it will be able to be seen.

Specifically, Lehman's (*syntonic* comma, 2004) and Zapf's solutions are virtually identical when the last version is transposed one place around the circle of fifths.

In the one hand, Dent applies an approximation to Zapf scheme in order to adapt its representation to the more usual description based on interval ratios according to their tempered fifths in relation to the values of commas.

Dent works out the average tempering produced in fifths with 1 beat per second and with 0.5 beats per second.[132] The result obtained is 3.7 cents. This distance is a very good approximation to either 1/6 of a *syntonic* comma (3.6 cents) or the scheme of the "55-division" of the octave, where the tempering is 3.8 cents. Likewise, the average in the fifths with 0.5 beats per second is 1.7 cents, which is also very close to 1/12 of a *syntonic* comma (1.8 cents).

According to the established approximation by fractions of a comma for temperaments defined by beat rates,[133] a theoretical tempering unit can be defined as 1/13th of a Pythagorean comma (approximately 1.8 cents) in Zapf's temperament. Then, Zapf's temperament can be approximated by a so-called "theoretical Zapf temperament"[134] which layout is shown in the following table:

C	G	D	A	E	B	F#	C#	G#	Eb	Bb	F
2/13	2/13	2/13	0	0	0	1/13	1/13	1/13	1/13	1/13	2/13

Another approximation is applied to Lehman I temperament, which is the only temperament based on Bach's diagram and also on fractions of the *syntonic* comma. Dent approximates the *schisma* (1/11 of a *syntonic* comma) to 1/12 of a *syntonic* comma obtaining a division of the comma in 13 identical units. Thus Lehman I temperament is approximated by a "theoretical Lehman I temperament"[135] which is based on the same theoretical unit which was obtained in the previous approximation of Zapf's temperament, that is 1/13 of a Pythagorean comma. Its layout is shown in the following table:

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[130] *Vid.* Section "Temperaments and key signatures in Bach's works".

[131] DENT, Thomas: "Zapf, Lehman and other Clavier-Well-Temperaments". In: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. July 2005. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. File posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>, 7 pages. [June 2010].

[132] According to Zapf's instructions, the calculations are applied to fifths with roots within the range from c to h (in Helmholtz's pitch notation) and they have been applied using the standard pitch or 'Baroque pitch' (Cammerton), i.e. about C=250 Hz).

[133] *Vid.* "Appendix 5: Definition of temperaments by beat rates", section "Approximation by fractions of a comma".

[134] This name is given by Thomas Dent.

[135] This name is given by myself in accordance with Thomas Dent's terminology.

C	G	D	A	E	B	F#	C#	G#	Eb	Bb	F
2/13	2/13	2/13	2/13	0	0	0	1/13	1/13	1/13	0	2/13

After this, if Zapf's model is transposed to G, that is one fifth upper, the following result is obtained:

C	G	D	A	E	B	F#	C#	G#	Eb	Bb	F
2/13	2/13	2/13	2/13	0	0	0	1/13	1/13	1/13	1/13	1/13

If both temperaments "are considered simply as collections of intervals without reference to their absolute pitch", they are identical apart from a single note which results 1/12 of a *syntonic* comma higher in the Lehman I temperament. be compared to Lehman's model (at pitch). The effect of this difference "may be audible, but cannot be decisive, especially as the thirds affected are all moderately tempered". Almost any melodic or harmonic progression which has a distinctive sound in the Lehman I temperament can also be played in the original untransposed Zapf temperament with the same effect, only that the distinctive character of each key is transferred one place around the circle of fifths. The transposition may, though, be significant when considering the relationship of particular key-colours with particular works.

In the second section of the article, Dent establishes a general method to interpret the diagram on the basis of two immutable basic principles and also of several decisions which have to be taken to define a temperament.

The two immutable principles are:

a) There are three tempered fifths of one size and five tempered fifths of another size, and three pure fifths separating them; the remainder (denoted by R) is determined how much these two classes of fifth are to be tempered.
b) One class of tempered fifth is twice as far as the other from being pure.

The decisions to be taken are:

a) In which direction the diagram is to be read.
b) What is the starting note.
c) Which type of tempered fifth is tempered by 2 units and which by 1 unit.
d) What is the size of the tempering unit.

The most important of these decisions is the choice of *direction* and the choice of *which fifths are tempered more*. Having fixed these, the choice of unit is constrained so that the remaining fifth R is not very wide or narrow; and the choice of starting note is more or less fixed by the expectation that F or C should have the purest major third.

The calculation of the tempering of the remaining fifth is the difference between the number of parts in which the comma is divided and the sum of all units in which the rest of tempered fifths of the circle are tempered.

According to this criterion, the decisions taken for each of the previous temperaments are:

a) Zapf temperament:
    a. The diagram is read from left to right.
    b. C is the starting note.
    c. 1/13 of a Pythagorean comma is the tempering unit (approximated by 1/12 of a *syntonic* comma).
    d. It assigns 2 units of temperament each to the three tempered fifths on the left and 1 unit each to the five on the right. The total is 11 units and the tempering of the remaining fifth is 2 units: R=13-11=2.
b) Lehman I temperament:
    a. Reads the diagram from right to left.
    b. F is the starting note.

c. 1/13 of a Pythagorean comma is the tempering unit (approximation of 1/12 of a syntonic comma).
d. It assigns 2 units of temperament each to the five fifths on the right and 1 unit each to the three on the left. The total is 13 units and the tempering of the remaining fifth is 0 units, that is an extra pure fifth: R=13-13=0.

Taking into account both most important decisions to be taken, Dent defines several new interpretations of Bach's diagram. There are four possible combinations in deciding which direction to read and which fifths are tempered more. Nevertheless, Dents only presents two of them, taken two different options for the size of the tempering unit in the first case and giving three new temperaments. Their skills are set out bellow:

1) Dent I temperament:
   a. The diagram is read from left to right.
   b. The three fifths on the left are tempered by 1 unit. Thus the five fifths of the right are temperd by 2 units. The total is 13 units.

   Consequently:

   c. Choosing a tempering unit of 1/12 of a *syntonic* comma (approximated by 1/13 of a Pythagorean comma),[136] the tempering of the remaining fifth is 0 units, that is an extra pure fifth: R=13-13=0.
   d. B is the starting note, in order to get C and F the best major thirds.

   It is also noticeable that the least pure major third occurs on C# and the least pure minor third on F.

   To sum up, Dent I temperament has the following structure:

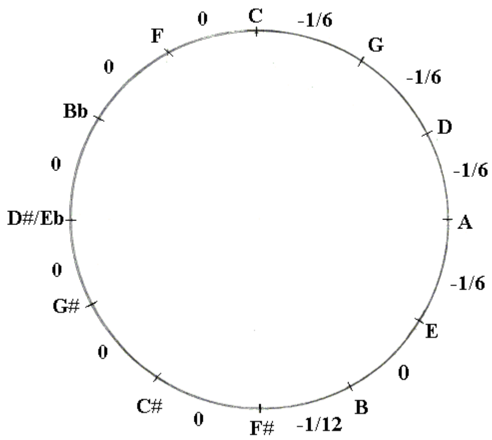

Figure 36 - Dent I temperament

2) Dent II temperament:

   Taking the same decisions as for Dent I temperament, if 1/15 of a Pythagorean comma is chosen as the tempering unit[137] (just under 1.6 cents):

   a. The tempering of the remaining fifth is 2 units: R=15-13=2.
   b. E is the starting note, in order to get C and F the best major thirds.

   It is also noticeable that the least pure major third lies on F# and the least pure minor third on Bb.

   To sum up, Dent I temperament has the following structure:

[136] This is the simplest possibility and "personal taste and judgement enter here".
[137] A "more abstruse possibility".

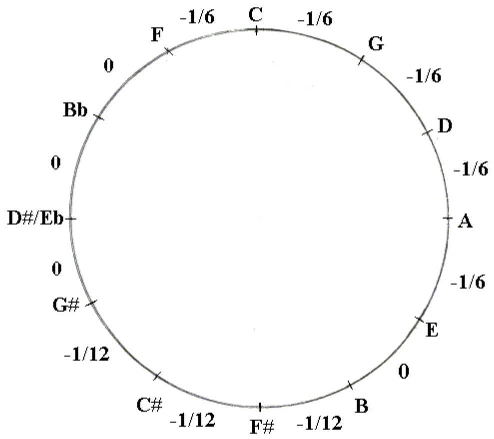

Figure 37 - Dent II temperament

3) Dent III temperament:
    a. The diagram is read from right to left.
    b. The five fifths on the right are tempered by 1 unit. Thus the three fifths of the left are temperd by 2 units. The total is 11 units.

    Consequently:

    c. Choosing a tempering unit of 1/12 of a *syntonic* comma (approximated by 1/13 of a Pythagorean comma),[138] the tempering of the remaining fifth is 2 units, that is an extra pure fifth: R=13-11=2.
    d. A is the starting note.

    To sum up, Dent III temperament has the following structure:

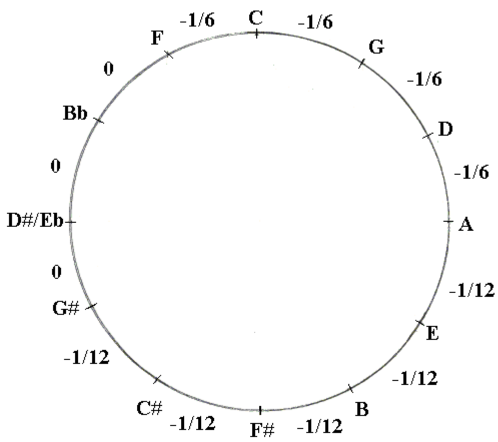

Figure 38 - Dent III temperament

It can be noticed that, after checking the sizes obtained for the thirds, the disposition of accidentals and progression of key-colours are similar for all Dent's temperaments and also closer to the 18th century norms, including those by Neidhardt and Sorge. Nevertheless, they are different from some other temperaments, like those by Zapf, Lehman I, Jencka or Francis.

Moreover, the widest thirds are placed in the farest tonalities, as happens for dozens of other Baroque circular temperaments. Here, the widest major thirds are placed on F#, C# and Ab and this is in contrast, for example, to Zapf temperament, where the

[138] 1/12 of a Pythagorean comma could be chosen but that would make half the tuning identical to Equal Temperament.

'worst' intonation occurs in D major, or to Lehman I, where it occurs in A major. This also occurs in Vallotti-Tartini (E/B major), Barnes (B major), Werckmeister III (B major) and most violently in Kirnberger III (A major).

David Ponsford (2005)[139] [Ponsford I & II]

David Ponsford, as an answer to Bradley Lehman's article,[140] says that the temperament proposed by Lehman "is good indeed, just as good as a number of other temperaments" but he doesn't agree with certain assertions made by Lehman.

Ponsford sees no reason to read the diagram downwards and, consequently, he reads it upwards. In his opinion, if the diagram had to be read downwards, Bach would have also written the 'C' upside-down. He also notes that the initial 'casual flourish' looks like a 'G' and this makes to think that G is the starting note. In this case, C is placed at the end of the diagram and this is also confirmed by the 'C' further along the line. The resulting sequence of notes throughout the circle of fifths is: G-D-A-E-B-F#-C#-G#/Ab-Eb-Bb-F-C. Nevertheless, Ponsford agrees that "the normal amount of tempering" is 1/6 comma although other possibilities could be considered.

After this, depending on the relationships between loops, loops with single knots and loops with double knots, that is, whether single knots represent fifths tempered by 1/6 or 1/12 comma, and double knots *vice versa*, two possibilities of tuning can be taken into account. This produces a temperament that is either similar to Lehman's (though not exact) or radically different. The sctructure of both resulting possibilities are shown in the following figures:

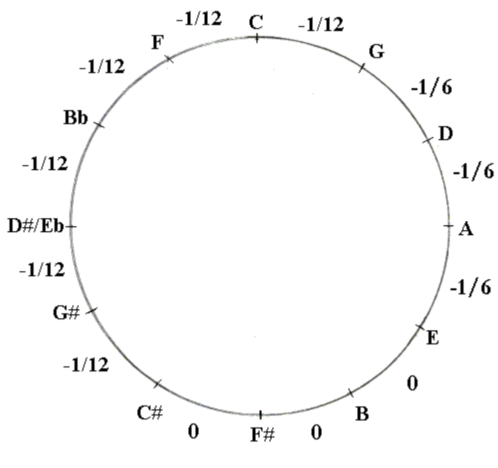

Figure 39 - Ponsford I temperament

[139] PONSFORD, David: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), p. 545.

[140] LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

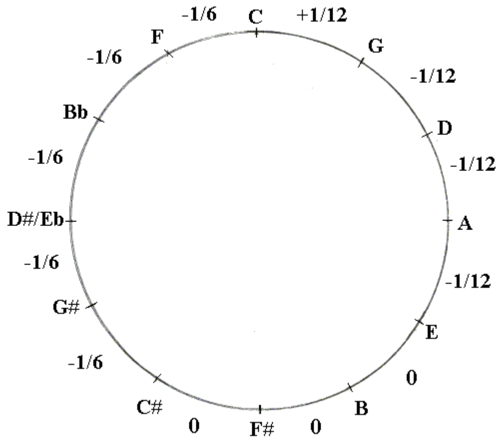

Figure 40 - Ponsford II temperament

Daniel Jencka (2005)[141]

Daniel Jencka, as an answer to Bradley Lehman's article,[142] sets out that the 'Rosetta Stone' can be interpreted in more than one way. He proposes an alternative reading which results in a somewhat different circulating temperament than that derived by Bradley Lehman.[143] He also makes reference to his essay and suggests its reading.[144]

Richard Maunder (2005)[145] *[Maunder I, II & III]*

Richard Maunder, as an answer to Bradley Lehman's article,[146] agrees that the Bach's diagram can be interpreted as a method of tuning but he does not think that Bach intended Bb to be lowered by 1/12 of a Pythagorean comma. This causes that the interval Bb-F is wider by 1/12 of a comma and makes the tuning of various intervals worse that they would be without it, as occurs in Lehman's solution. Sloan thinks that the end of the diagram would have to be interpreted as a plain loop instead od a single knot, giving a pure fifth between Bb and F.

Sloan suggests various other ways of interpreting Bach's diagram giving three new temperaments according to the meaning of each kind of loop:

a) Double knot = 1/6 comma, single knot = 1/18 comma, single loop = pure.
b) Double knot = 1/6 comma, single knot = 1/24 comma, single loop = 1/24 comma.
c) Double knot = 1/7 comma, single knot = 1/14 comma,. Single loop = 1/14 comma.

The structure of each of the previous temperaments is shown in the following figures:

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141 JENCKA, Daniel: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), p. 545.
142 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
143 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
144 *Vid.* Section "Daniel Jencka (2005)".
145 MAUNDER, Richard: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), pp. 545-546.
146 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

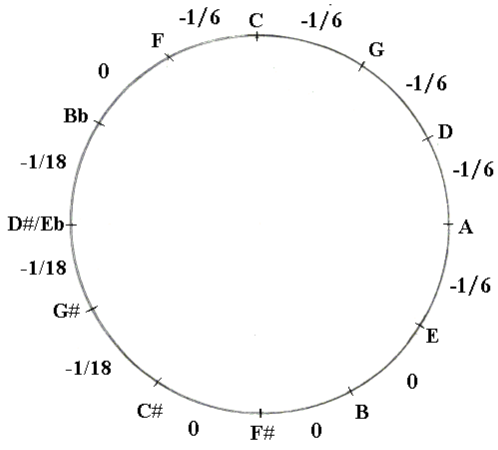

Figure 41 - Maunder I temperament

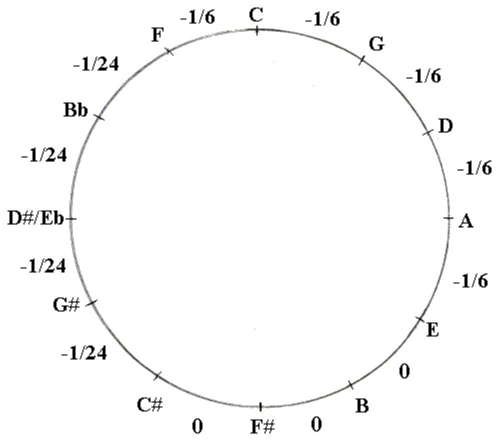

Figure 42 - Maunder II temperament

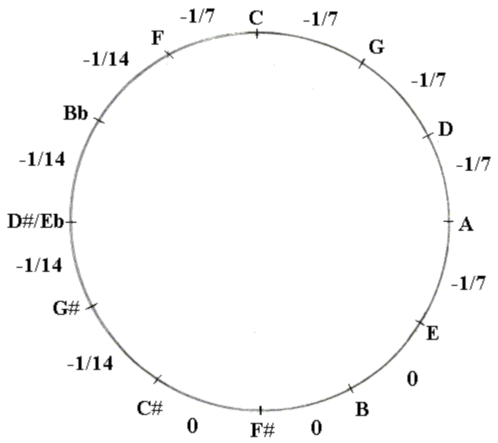

Figure 43 - Maunder III temperament

Sloan temperaments don't differ very much from Lehman's interpretation except in the placing of Bb but he concludes that "Bach's diagram on its own is insufficient to define one unique temperament precisely.

Carl Sloan (2005)[147]

Carl Sloan, as an answer to Bradley Lehman's article,[148] thinks that the 'C' indicating the starting-point of the temperament is actually an ornamental hook on the 'C' of 'Clavier' since the same hook is also found in other places without any other meaning. It appears, for example, in the title-page of the second volume of *The Well-Tempered Clavier* (not in Bach's hand and where there are no loops) and on the initial 'S' of 'Semitonien' and 'Sebastian'.

Sloan adds that Lehman is clearly unaware that the precise relation between beat rates and tempering was not understood in 1722. Without a knowledge of beat rates, neither the accuracy nor reproducibility required by Lehman's temperament would have been attainable.

Mark Lindley (2005)[149]

Mark Lindley, as an answer to Bradley Lehman's article,[150] desagrees with its main conclusion and announces a further article in which he plans to explain the reasons in collaboration with Ibo Ortgies.[151]

Kenneth Mobbs and Alexander Mackenzie of Ord (2005)[152] *[Mobbs-Mackenzie]*

Kenneth Mobbs and Alexander Mackenzie of Ord, as an answer to Bradley Lehman's article,[153] set out some points in which they are in disagreement with Bradley Lehman:

1) Neither Sparschuh nor Zapf were acknowledged in Lehman's two printed articles.[154]
2) The inversion of the diagram is a wrong idea.
3) The little 'c' placed on top of the capital letter 'C' of 'Clavier' is merely decoration since it is separated from the body of the capital letter.

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147 SLOAN, Carl: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), p. 546.
148 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
149 LINDLEY, Mark: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), p. 546.
150 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
151 LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning". In: *Early Music*, Vol. 34, No. 4 (November 2006), pp. 613-623.
152 MOBBS, Kenneth & MACKENZIE OF ORD, Alexander: "Tempering Bach's temperament [correspondence]". In: *Early Music*, Vol. 33, No. 3 (August 2005), pp. 546-547. Also available in MOBBS, Kenneth & MACKENZIE OF ORD, Alexander: "Copy of Early Music, Vol XXXIII no. 3 (August 2005), Correspondence, pp. 546 – 547. © Oxford Journals, Oxford University Press". In: *Kenneth Mobbs Early Keyboards* [online]. August 2005. In: <http://www.mobbsearlykeyboard.co.uk/EarlyMusicAug.2005KWMACNMletter.htm>. [April 2011]. An amplification of the previous letter was drawn up: MOBBS, Kenneth: "The 'Bach Temperament'. Kenneth Mobbs' amplification of the letter he and Alexander Mackenzie of Ord published in the August 2005 edition of *Early Music*, (pp.546-7)". In: *Kenneth Mobbs Early Keyboards* [online]. 28 August 2005. In: <http://www.mobbsearlykeyboard.co.uk/KWMamplificationofEMAug05letter.htm>. [April 2011].
153 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*
154 LEHMAN, Bradley: "Bach's extraordinary temperament: our Rosetta Stone – 1"..., *op. cit.*; LEHMAN, Bradley: "Bach's extraordinary temperament: our Rosetta Stone – 2"..., *op. cit.*

4) The 1/6-comma tempering is extremely complicated and Lehman does not provide either any evidence for this value or any details as to how it should be. He constantly refers to “normal” experience.

Mobbs and Mackenzie thinks that it is very difficult to temper one interval by 1/6 of a Pythagorean comma *by ear alone*, particularly so if one does not make use of reference beatings from other notes already tuned, for instance from those notes in a chain of pure fifths. They also argue that the majority of all unequal temperaments at Bach's time were based on the tuning of initial intervals as in mean-tone temperaments and they would be equipped to construct a chain of four 1/4-comma fifths making a pure third.

According to this, they suggest other interpretation of the diagram which is based in the following basic principles:

a) The diagram is read upwards starting from the left-hand side.
b) C is the starting note since this is the conventional keyboard tuning.

This is the explanation of their method if tuning:

1) From C, tune three normal mean-tone fifths: C-G-D-A.
2) From A, tune three pure fifths: A-E-B-F#.
3) From F#, tune five equally-flat but very slightly tempered fifths: F#-C#-G#-Eb-Bb-F, that is, narrowing them almost imperceptibly, so that the remaining fifth F-C is also judged to be of the same size as its immediate predecessors and the tuning circle is closed.

The sctructure of the resulting temperament can be shown in the following figure:[155]

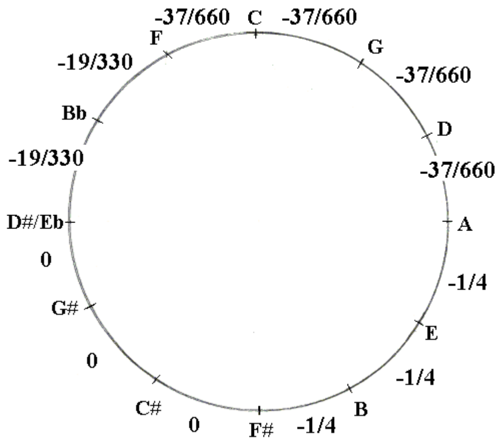

Figure 44 - Mobbs - Mackenzie temperament

Stuart M. Isacoff (2005)[156]

Stuart M. Isacoff answers to some comments made by Bradley Lehman[157] regarding his book.[158]

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[155] Mobbs an Mackenzie do not give any exact ratio for the "very slightly tempered fifths". The values of the figure calculated are given in: LEHMAN, Bradley: “Other ‘Bach’ temperaments”…, *op. cit.*

[156] ISACOFF, Stuart M.: “Tempering Bach’s temperament [correspondence]”. In: *Early Music*, Vol. 33, No. 3 (August 2005), p. 547.

[157] LEHMAN, Bradley. “Bach’s extraordinary temperament...”, *op. cit.*

[158] ISACOFF, Stuart M.: *Temperament*. New York: Alfred Knopf, 2001, 277 pages. Dutch translation by Henk Moerdijk: *Het octaaf. De juiste stemming in de muziek.* Amsterdam: De Bezige Bij, 2002, 271 pages.

Early Music Editor (2005)[159]

The editor of *Early Music*, as an answer to Bradley Lehman's article[160] and taking into account the letters received about it, gives more information about other works which can be found on the. Likewise he also makes reference to lots of letters to *Early Music* which have not been published. He quotes the following authors: John Charles Francis, Keith Briggs, Bradley Lehman, Michael Zapf and Andreas Sparschuh.

Bradley Lehman (2005, 2)[161]

This article by Bradley Lehman is "an elaboration and exploration of some of the principles presented in 'Bach's extraordinary temperament: our Rosetta Stone'.[162]

Lehman II temperament "gives an excinting, colourful sound and has the flexibility to play equally well in all keys, with no dead-ends anywhere. 'Equally well' is not 'the same', however. All those major and minor keys sound objectively *different from* one another, having slightly different semitone and tone arrangements in their scales, and different harmonic balances of the triads and more complex chords. These differences render all the keys recognizably distinct, as to the tensions and resolutions in the way tonal music behaves." Nevertheless, "we have the complete flexibility of modulation as if it were equal temperament." This temperament also reveals marvelous colors with dramatic and beautiful constrats that are never harsh.

Speaking more specifically about the clavichord, one has to take into account that its notes have a more important attenuation, that is to say, they die away more quickly, compared with the hasrpsichord or organ. The consequence is that its tuning is more difficult to hear accurately. Other elements make also more difficult to hear the tuning of the clavichord like the expressive pitch-bending with the fingers and the soft volume of the instrument.

Depending on the music to be played and for equal finger pressure, the following factors have to be taken into account in the tuning process:

1. Flexibility to play all the music we care to play (as appropriate to the style and compass of the instrument).
2. Euphonious results, at least avoiding grossly noticeable errors of intonation.
3. Easy of tuning.
4. The need not to retune too often –which becomes especially bothersome on clavichords that are double-strung, and potentally destructive to the instrument if it is double- or triple-fretted.

After this, Lehman gives a specific practical method for tuning the clavichord, method which he has tried in his own instrument.[163] Several clavier works by Johann

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159 [*Early Music*'s editor]: "Tempering Bach's temperament [correspondence, editor's note]". In: *Early Music*, Vol. 33, No. 3 (August 2005), pp. 547, 548.

160 LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

161 LEHMAN, Bradley: "The 'Bach temperament' and the clavichord". In: *Clavichord International*, Vol. 9, No. 2 (November 2005), pp. 41-46. Also available in: "The 'Bach temperament' and the clavichord". In: *Johann Sebastian Bach's tuning* [online]. October 2006. In: <http://www-personal.umich.edu/~bpl/larips/clavichord.html>. [June 2010].

162 *Vid.* Section "Bradley Lehman (2005, 1) [Lehman I & II]".

163 Other versions of this practical method of tuning by ear are set out in several sources: LEHMAN, Bradley: "Practical temperament instructions by ear"…, *op. cit*; and LEHMAN, Bradley: "A quick and practical bearing method, by ear"…, *op. cit.*Nevertheless, according to the author, the method set out in this article is the simplest way.

Sebastian Bach demonstrate that whatever instrument these compositions are played on, any proposed retuning before or during the music is out of the question. This layout also allows him to play all the 18th-century music and, even, 16th- and 17th-century music also sounds pleasant. The main basis for this affirmation is in the existence of several augmented or diminished intervals, and also the existence of enharmonic notes as a result of the modulations or other nonchord tones. Moreover, more of these enharmonic ones occur in accented thematic positions of the pieces.

Each time 12 notes are traversed in any 12-tone temperament, a Pythagorean comma is gained or lost. Thus any composition that exceeds 12 notes has some wrong notes by the distance of this comma. The simpler solution, instead of retuning, is to have a temperament that gains or loses this comma gradually as occurs in Lehman II temperament. So there is never any obtrusive effect.

George Lucktenberg (2005)[164] [Lucktenberg]

After reading Bradley Lehman's article,[165] George Lucktenberg proposes a more simple temperament which is also an interpretation of the squiggle but it is more easy to teach to students. Lehman's model "requires a distinction between a 1/6 and a 1/12 comma" and consists "essentially in an "8 and 4" (eight fifths tempered, four pure)", a model which is "persuasive if not wholly provable." Because of this, Lucktenberg defines a model with only one kind of tempered fifth.

This is a very practical temperament since it is defined as a method of tuning. The inconvenient is that he does not give any exact mathematical definition for his proposition. This is the process decribed by him:

1) Middle C from fork or other source;
2) Tune the C one octave bellow Pure (hereinafter, "P");
3) Middle C down to F, Tempered (hereinafter, "T");
4) F up to Bb, P;
5) Bb down to Eb, T;
6) Eb up to Ab, P;
7) Ab down to Db, T;
8) Db up to F#, P.

Next, strarting with the lower C (NOT middle C!):

9) Up to G, T;
10) G down to D, T;
11) D up to A, T;
12) A down to EW, T;
13) E up to B, T.

The B down to F# should be pure at that point; if not, you might "backtrack" to see if one or another tempered fifth or fourth may be too wide or narrow.

The result is one temperament with an unique kind of tempered fifth in F-B, C#-G# and D#/Eb-Bb. The rest of the fiftths are pure. Lucktenberg doesn't give any exact value fo the amount of tempering. Thus, setting 1/8 of a Pythagorean comma for the tempering value on average,[166] the structure of Lucktenberg temperament would be approximately like this:

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164 LUCKTENBERG, George: "Light Reading for the Winter". In: *Southeastern Historical Keyboard Society (SEHKS) Newsletter*, Vol. 26, No. 1 (December 2005).

165 LEHMAMN, Bradley: "Bach's extraordinary temperament...", *op. cit.*

166 LEHMAN, Bradley: "Other 'Bach' temperaments"…, *op. cit.*

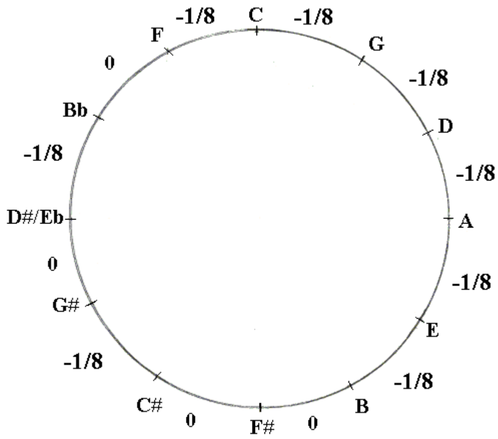

Figure 45 - Lucktenberg temperament

As Lucktenberg says, this temperament "is remarkably smooth through all keys, while preserving the usual escalation of beats in the major thirds of keys with the most sharps and lats in the signature, and the usual 'color'-change from key to key." He also considers that it is "a practical alternative either satisfactory for works of J. S. Bach and successors in the mid-to-late eighteenth century."

Daniel Jencka (2006)[167] [Jencka]

Daniel Jencka, as an answer to Bradley Lehman's article,[168] yields a slightly different interpretation of Bach's diagram in relation to the solution given by Lehman. His inquiry begins "with the conjecture that the somewhat curious single loop floating off at the end of the script also signify a (final) pure fifth, specifically from Bb to F." The question is to prove if the previous suggestion results in a workable temperament.

If the fifth Bb-F is pure and if the five fifths between F and E are narrowed by a sixth of a Pythagorean comma, the remaining amount of a sixth of a Pythagorean comma would have to be absorbed by the three fifths between C# and A# (Bb). The most logical thing would be to distribute this excess uniformly narrowing each fifth by 1/18 of a comma. Nevertheless, there is no historical temperament which uses 18ths of a comma.

One temperament of this kind could be devised taking a common modified meantone approach and beginning with a set of consecutive fifths reduced by 1/6 of a Pythagorean comma on the white keys. Thus if one of these sixths of a comma is redistributed between three fifths, the result is a "fine, "well" temperament."

Starting with the fifths F-C-G-D-A-E-B tempered by -1/6 of a comma, a good potential 1/6 of a comma to redistribute would be that of E-B, thereby the resulted sequence of fifths is constructed on tones sensibly relating to open strings on violin family instruments.

An ideal circulating temperament would have gradually widening major thirds as one move away from C in either direction and this typically led to a wide and wild major third peak at the most remote keys, typically on C# or F.

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[167] JENCKA, Daniel: "The Tuning Script from Bach's Well-Tempered Clavier: A Possible 1/18th PC Interpretation". In: *1/8th PC WTC Tuning Interpretation* [online]. 2 June 2006, revised after original essay of 3 March 2005. In: <http://bachtuning.jencka.com/essay.htm>. [June 2010].
[168] LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

Making the fifths E-B, G-B and B-F# pure, a desired gradual increase in the size of the thirds is yielded and making Bb-F pure, a gradually larger major third on Bb-D is also yielded. The notes G# and D#/Eb are the typical starting points in a modified meantone temperament. Finally, the fifths between C# and Bb can be barely narrowed the remaining amount of 1/6 of a Pythagorean comma and this results in a circulating temperament. The resulting value for these fifths is -1/18 of a Pythagorean comma.

It is easy to note that major thirds grow consistently wider in both directions away from C up to the peak at E-G#, with no jumps in size anywhere. Thus the condition for an ideal temperament is carried out.

In this temperament, 1/18 of a comma is the smallest quantity represented and it is easy to note that the amount of 1/6 of a comma can be represented as 3/18 of a comma, that is the triple.

The structure of the resulting temperament can be shown in the following figure:

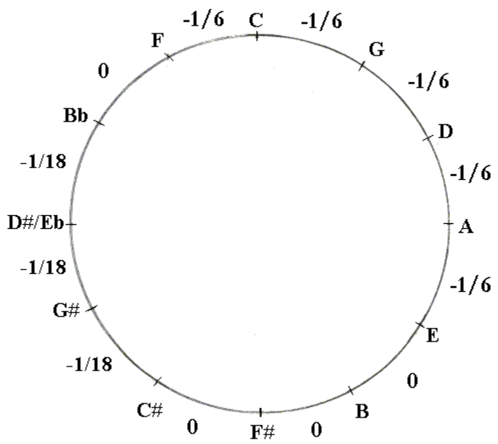

Figure 46 - Jencka temperament

Going back to the scroll, some skills of the graphic can be useful to verify the previous scheme. There are three loops with single squiggles on the left of the diagram, three empty loops, five loops with more elaborated or double squiglles and a single empty loop floating off the very tail of the script. The single and double squiggles inside the larger loops are of two legible sizes. The pairs of squiggles inside the set of five loops are consistently legible as being of two sizes, showing a two-to-one relationship. Moreover, the single squiggles inside the set of three loops are small, like the smallest squiggles inside the set of five loops. None of the inner loops on the left is larger than the innermost loops of the figures on the right.

This means that two easily distinguishable sizes of quantifying loops are used to represent a smaller and a larger quantity, appearing either singly or together inside any fifth loop. Specifically, an easy interpretation is a ratio of 1 to 2. Thereby, the smallest inner loop would equal 1/18 of a comma and any larger inner loop would equal 2/12 of a comma. Thus both loops together represents the addition of their respective values, that is to say:

$$1/18 + 2/18 = 3/18 = 1/6$$

It is easily deductible mathematically that the single empty loop floating off the very tail of the script represents a pure fifth corresponding to the interval Bb-F.

Bradley Lehman (2006, 1)[169] [Lehman III]

A new simple temperament is formulated by Bradley Lehman that takes "two or four minutes to set up in the bearings area". It emphasizes the interval quality instead of the maths or any kind of counting. There is basically one size of fifth in this temperament and no notes need to be retuned after they have been put into place. Lehman gives the practical instructions to tune the instrument and also gives some examples of works by Johann Sebastian Bach and François Couperin which can be used in order to test its quality and usability. Later, Lehman posted a more detailed description and evaluation in his response to an article by John O'Donnell.[170] The structure of this temperament, which can be identified by Lehman III, is given in the following figure:

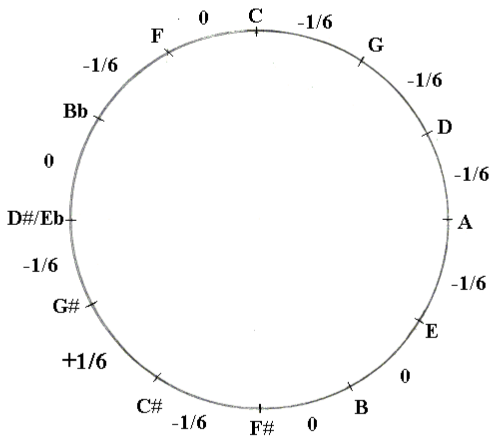

Figure 47 - Lehman III temperament

According to the author, "there's an odd set of properties [of this temperament] that come up if the intial C-E is too wide and venturing into the blur-range:"

- The natural keys sound as if they're reluctant to relax, noticeable especially in F major; and the Bb major triad gets difficult to set with any reasonable quality. C, F, G, and Bb are still the best four, but we've lost a good bit of their resonance. The D major triad also emerges sounding rather hectic.
- Around the service entrance at the back of the building, the leftover C# to G# 5th (or C# to Ab) turns out narrow.
- The triads of Db major, B major, and F# major turn out much too consonant, ahead of the qualities of D/Eb, A/Ab, and E. It's as if the shape flips itself inside-out: making the key of C# major sound fantastic, but at the expense of too much "nervous tension" everywhere else.

This temperament is based on the "Bach's 1722 diagram" but it is also based in some properties of O'Donnell's temperament since it is design as a response to his

[169] LEHMAN, Bradley: "Another easy circulating temp, emphasizing tasteful adjustment". Posted to: *LISTSERV HPSCHD-L: Harpsichords and Related Topics* [online]. 8 March 2006. Group HPSCHD-L@LIST.UIOWA.EDU in <http://list.uiowa.edu/scripts/wa.exe?A2=ind0603&L=HPSCHD-L&T=0&O=D&P=225985>. [December 2010]. LEHMAN, Bradley: "Bach's temperament, Occam's razor, and the Neidhardt factor". In: *Johann Sebastian Bach's tuning* [online]. November 2006. In: <http://www-personal.umich.edu/~bpl/larips/lindleyortgies.html> & <http://www.larips.com>. [December 2010]. Response to: O'DONNELL, John: "Bach's temperament, Occam's razor...", *op. cit.*

[170] *Vid.* section "Bradley Lehman (2006, 4)".

article. Nevertheless, Lehman's temperament differs from O'Donnels in several questions.[171] Its main principles are:

- There are only two different qualities of fifths and all tempered fifths have the same amount and quality as one another sounding identical in musical practice, even if one of those might be in the opposite direction.
- The whole line is moved over by one note so the big D of the diagram means D (instead of D#/Eb).
- The drawing is a chromatic sequence instead of a series of fifths.
- The diagram is read upwards.

Thus the diagram is interpreted in the following way:

- The layout starts on C as the first large loop of Bach's drawing at the left.
- The tiny loop to its left is the pure B-F#.
- The third large loop indicates the note D and the fifth D-A (not D# as O'Donnell's layout has it).

Following the chromatical order, the sequence of intervals is:

- B-F# corresponds to the end of the diagram and it tuned as pure.
- The three loops with single squiggle correspond to the intervals C-G, C#-G# and D-A- They are all tempered in the same amount / quality as one another, sound identical in musical practice, even if one of those might be in the opposite direction. Finally, the fifth C#-G# is tuned wider than pure by the same amount.
- The three simple loops correspond to the fifths Eb-B-b, E-B and F-C and they are tuned as pure.
- Finally, the rest of loops, those with double squiggles, correspond to the fifths F#-C#, G-D, G#-D#, A-E and Bb-F. They are all tempered the same amount as one another and the same as the first set.

Finally, eight tempered fifths and four pure are obtained. All tempered fifths are reduced by the same amount except C#-G# which is enlarged by the same quantity. The amount is defined as 1/6 of a Pythagorean comma but it can be slightly different (1/7 or 1/8 of a Pythagorean comma). The difference between the loops with single squiggles and the loops with double squiggles is that the ones at the left are tuned as widened fourths bellow, and the ones at the right as narrowed fifths above.

Bradley Lehman (2006, 2)[172]

This article by Bradley Lehman is a simpler explanation and justification of Lehman II temperament. Lehman starts with the suggestion that the decorative scroll of the title page of Bach's personal copy of *The Well-Tempered Clavier* means something in relation to the temperament and he continues with the description of the method of tuning.

The diagram starts with the fifths F-C-G-D-A-E, the diatonic notes of the C major scale, using the common practice of medium tempering with a consistent size: two nudges from pure, that is to say, -1/6 of a Pythagorean comma. As indicated by the first empty loop on the duiagram, the fifth E-B simply remains a pure fifth, with no nudges. The five accidentals are found in the B major scale and these notes are arranged by fifths in the following sequence E-B-F#C#-G#-D#-A#. The notes E and B are the common notes in C and B major scales. Now the aim is to make the rest of the notes

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[171] *Vid.* section "Bradley Lehman (2006, 4)".

[172] LEHMAN, Bradley: "In Good Temper". In: *BBC Music Magazine*, Vol. 14, No. 13 (August 2006), pp. 42-44.

tastefully irregular in a way that improves their utility. Continuing with the sequence, the fifths E-B-F#-C# are pure since they correspond to the three consecutive plain loops on the diagram. After this, the next G# is made pure from the C# and the major third E-G# becomes so wide and harsh that it sounds rough. The remaining fifths (C#-G#-D#-A#) are tempered one nudge each, that is to say, -1/12 of a Pythagorean comma. Finally, the leftover interval of A# back to F is not tuned directly; it is very nearly pure anyway, that is, enlarged by 1/12 of a Pythagorean comma, and "serves as a checkpoint that we have not ruined ourselves with cumulative errors."

The application of this method of tuning to the harpsichord obtains a result in which “all 24 major and minor keys become usable and distinctive, with a smooth variety passing through all modulations.”

Mark Lindley & Ibo Ortgies (2006)[173] [Lindley-Ortgies I & II]

Mark Lindley and Ibo Ortgies, as an answer to Bradley Lehman’s article[174], proposed two more interpretations of Bach's diagram. Moreover, they make reference to two other solutions given by Daniel Jencka[175] and Thomas Glueck,[176] as well as Sorge and Neidhardt’s temperaments.

Lindley and Ibo Ortgies start with a review of Lehman's article showing some points which they are or not in agreement with.

They agree with the fact that thirds among the naturals (F-B) have to be tempered less than those involving sharps or flats while the tempering of the other thirds tend to vary gradually in keeping with their relative places around the circkle of fifths.

Lindley and Ibo Ortgies don't think that the small letter 'c' near the 'C' of 'Clavier' means something in relation to the temperament. The 'small c' is a kind of serif applied frequently in Bach's handwritting to certain capital letters, especially C, S, E, F and sometimes K.

They also agree that Sorge was relatively closely associated with Bach but they don't think that Sorge or Neidhardt tempered the third E-G# as much as Lehman says Bach always tempered it, that is, a Pythagorean third reduced by 1/12 of a Pythagorean comma or a just third enlarged by 1/4 (3/12) of a Pythagorean comma. Neidhardt would never temper E-G# more than Ab-C either.

There are no numbers in the scroll and it is unwarranted to assume that Bach must have tempered some fifths exactly twice as much as others. It can be assumed hypothecally that each kind of loop indicates merely a uniform tempering of the fifths. This the following assumptions can be assumed:

a) Loops with an elaborated internal squiggle which refer to F-C-G-D-A-E[177] must of course be slightly smaller than in equal temperament instead of fifths tempered by exactly two theoretical units.
b) Simple loops, which refer to the fifths E-B-F#-C#, might refer to one size of fifth which could be considered standard in Bach's day, not necessarily pure.
c) Loops with a simple squiggle, which refer to the fifths, C#-G#-D#/Eb-Bb, indicate a different kind of relation to the standard type of fifths represented

[173] LINDLEY, Mark & ORTGIES, Ibo: “Bach-style keyboard tuning”. In: *Early Music*, Vol. 34, No. 4 (November 2006), pp. 613-623.
[174] LEHMAN, Bradley. “Bach’s extraordinary temperament...”, *op. cit.*
[175] *Vid.* Section "Daniel Jencka (2006) [Jencka]".
[176] *Vid.* Section "Thomas Glueck [Glueck]".
[177] According to Bradley Lehman.

by the simple loops. They might indicate fifths slightly larger than the standard type.

Taking into account these principles, Lindely and Ortgies give an alternative interpretation of Bach's diagram which don't involve any gradations finer than half of the arbitrary unit. This arbitrary unit corresponds to a *schisma*, a unit already used by Neidhardt and Sorge.[178] The assigned values for each kind of fifth are corresponding to each kind of loop are:

a) Single loops correspond to intervals tempered (reduced) by one theoretical unit of impurity.
b) Loops with single spiral correspond to intervals tempered (reduced) by 1/2 unit.
c) Loops with double spiral corresponds to intervals tempered (reduced) by 1½ unit.

Thereby, the double spiral corresponds to a greater amount of impurity than the single spiral, as it happens in Lehman's scheme. The mean difference is in the fact that single loops correspond to a mean amount and not to a null amount. Thus this interpretation of the scroll does not consider the existence of pure fifths with the exception of the interval represented by the extremes of the scroll.

Likewsie, the calculation of the tempering of the major thirds would show that the third E-G# is tempered less than G#/Ab-C.

The structure of the temperament in question, which can be identified as Lindley-Ortgies I temperament, is shown in the following figure, where it is represented by the usual fractions of a Pythagorean comma:[179]

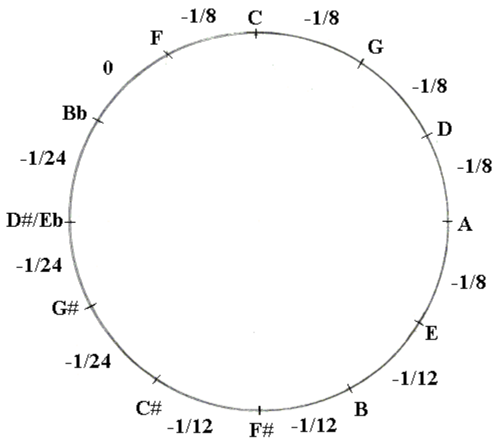

Figure 48 - Lindley - Ortgies I temperament

This temperament can be identified as. Another alternative is also given by Lindley and Ortgies. In this case, the loops are interpreted in the same way as Lehman, with regard to the amount of tempering which corresponds to each kind of loop. The difference is in that Lindley and Ortgies assume that the loop touched by the serif refers to F-C rather than to C-G as Lehman does. Really, this interpretation of the scroll is equivalent to a transposition of the scheme given by Lehman up a fifth. In this case, the thirds E-G# and G#/Ab-C are tempered by the same amount.

The structure of this temperament, which can be identifies as Lindley-Ortgies II temperament, is shown in the following figure:

[178] *Vid.* "Appendix 3: Representation of 'good' temperaments", section "Representation with schismata".

[179] *Vid.* also "Appendix 3: Representation of 'good' temperaments", section "Representation with schismata" for the equivalence between the two representations.

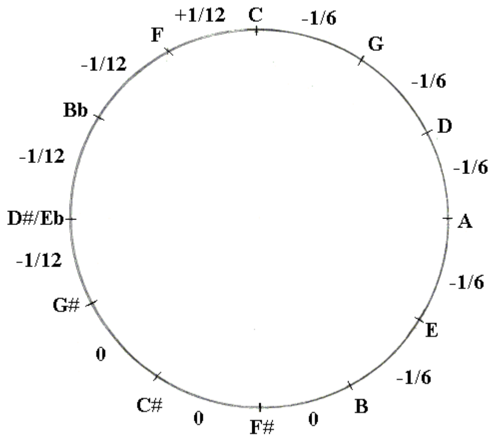

Figure 49 - Lindley - Ortgies II temperament

Bradley Lehman (2006, 3)[180]

According to Mark Lindley and Ibo Ortgies,[181] "no tuning-theorist close to Bach approved of tempering E-G# as much as Lehman does". Moreover, "Neidhardt would never countenance tempering E-G# more than Ab-C". They also makes reference to Sorge and Bach himself.

Bradley Lehman, on the other hand, says that Neidhardt did both these things in his 1732 - 5th circle XI temperament, that is, the third E-G# is tempered as widely as Lehman does and it is also wider than Ab-C. Lehman also says that, "in total, six of the 21 Neidhardt examples published in 1732 break one or both of the Lindley/Ortgies points about the positioning of the note G#/Ab from E or C on either side if it."

In their final version, Lindley and Ortgies change the last paragraphs argueing that this skills are only referred to the musical practice: "Neither Sorge nor Neidhardt ever countenanced tempereing E-G# (for any reason whatever) in actual musical practice by as much as Dr. Lehman says Bach always tempered it"; and "nor indeed did Neidhardt ever countenance tempering E-G# more than Ab-C in any tuning that he recommended for use in any kind of social context whatever (i.e. at a court, in a large city, in a small town, or in a village)."

Then Lehman argues that Sorge recommended his 1758 temperament for use especially in *Chorton* organs; the performance and improvisation of church music presumably being a situation that is both "actual musical practice" and a "social context". Moreover, the article by Lindley and Ortgies "doesn't stablish any rules by which we're supposed to assume that Neidhardt and/or Sorge were joking, or making up merely impractical speculations!".

According to Lehman, another question is problematic in the article by Lindley & Ortgies: their unchanged phrase "by as much as Dr. Lehman says Bach always tempered it". In response to this assertion, Lehman argues that his article "is principally about playing the *Well-Tempered Clavier* on harpsichords and clavichords, and not such "always" is any central part of my argument. And yet, Lindley and Ortgies seek to

[180] LEHMAN, Bradley: "Bach-style keyboard tuning". In: *Johann Sebastian Bach's tuning* [online]. October-November 2006. In: <http://www-personal.umich.edu/~bpl/larips/lindleyortgies.html> & http://www.larips.com>. [December 2010]. Response to: LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning"…, *op. cit.*

[181] *Vid.* LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning"..., *op. cit.*

knock this down through appeals to organ-tuning treatises, in social contexts where Bach's preludes and fugues were not the music to be played?"

With regard to the ornament near the 'C' of 'Clavier', Lehman thinks that "the "misreading a serif as a letter" *is not* a make-or-break point of my hypothesis (ans never was). My "misreading" of an object looking like a letter C was merely a *catalyst* for me to experiment with temperament layouts that have middle C in that particular position. But my primary reason is, and remains, my observation that the entire C major scale is tuned first on the keyboard, as a set of regular naturals (except B), which necessitates beginning the line with F; and C indeed falls into that position where that "serif" happens to look like a C. Coincidence?"

The article follows with "other problems of fallacies, topic-dodging, and tone in the Lindley/Ortgies article".

John O'Donnell (2006)[182] [O'Donnell]

John O'Donnell, also as an answer to Bradley Lehman's article,[183] proposes a different interpretation of the Bach's diagram. He also makes reference to Sorge and Neidthardt's temperaments.

O'Donnell interprets the scroll as an horizontal line rather than a circular diagram, as some previous contributors thought, that is to say, the diagram can be read chromatically. This assertion is supported by several arguments:

The chromatic order coincide with the order in which Bach sets out the preludes and fugues of *The Well-Tempered Clavier*.

Temperaments in Bach's day were commonly notated chromatically from C. This custom is influenced by the equal temperament which is a general model to measure the intervals, apart from the model of the monochord. Although some theorists like Neidhardt or Carl Philipp Emanuel Bach described their temperaments on the basis of fifths and fourths, Werckmeister, for instance, show a number of temperaments laid out on a monochord and O'Donnell thinks that this is indeed Bach's model.

The letter 'D' of 'Das' includes a diagonal line that goes directly to the third loop and has 'Eb' on the left side of the line joined to the German keyboard tablature version of 'Dis' (D#) on the other side of the line.

It may be that the choice of Eb/D# was made to symbolize the whole art of well-tempered tuning, wherein enharmonic notes are made one, and it is interesting in this regard that in the ensuing volume only the Eb minor Prelude and D# minor Fugue are notated as an enharmonic pair. If Eb/D# is identified in the diagram, it is logical to accept that C is placed on the left side of the scroll and, starting at this point, the rest of loops correspond to the rest of notes of the chromatic scale, that is, from C to B.

It can be also noticed that the flourish of the letter 'D' passes, by accident or design, through the middle of the loop now interpreted as D.

Working within the limits of the octave, each time we leap up a fifth we land on a loop enclosing a double coil, while each time we leap down a fourth we land en either a loop that encloses a single coil or an empty loop. Thereby, O'Donnell interprets the single and double coils as widened fourths and narrowed fifths respectively. The empty loops also represent fifths. This suggestion agrees with the fact that the single coils occur at the entrance to a loop while the double coils occur at the exit from a loop showing different phenomena.

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[182] O'DONNELL: "Bach's temperament, Occam's razor, and the Neidhardt factor". In: *Early Music*, Vol. 34, No. 4 (November 2006), pp. 625-633.
[183] LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

The values of the amounts of impurity of the different intervals are not specified by Bach but it can be assumed that the purer thirds should be placd in teh vecinity of C in a circulating temperament. The following statments about major thirds are set out by O'Donnell:

1. C-E and F-A will always be identical in size;
2. G-B will always be narrower then D-F#;
3. A-C# will always be wider than E-G#;
4. B-D# will always be wider than F#-A#;
5. F#-A# will always be narrower than Db-F;
6. Db-F will always b narrower than Ab-C; and
7. Bb-D will always be wider than F-A.

In order to decide the relative sizes of the intervals, and do not taking into intervals narrowed smaller than a 1/12 of a Pythagorean comma, two possibilities can be considered:

a) 2 fifths reduced by 1/4 of a comma and 6 reduced by 1/12 of a comma.
b) 4 fifths reduced by 1/6 of a comma and 4 reduced by 1/12 of a comma.

The first option can be ruled out since it can be demonstrated that it is not a workable solution. Since the thirds on F, C, and G -those which are expected to be closest to pure in a circulating temperament- are each generated by three narrow fifths and one pure fifth, and since the third on F# -normally a wider interval- is generated by four narrow fifths, it seems obvious to place the 1/6-comma narrowings in the region of C and the 1/12-comma narrowings in the region of F#.

The structure of the resulting temperament, which can be identified as O'Donnell temperamengt, is shown in the following figure:

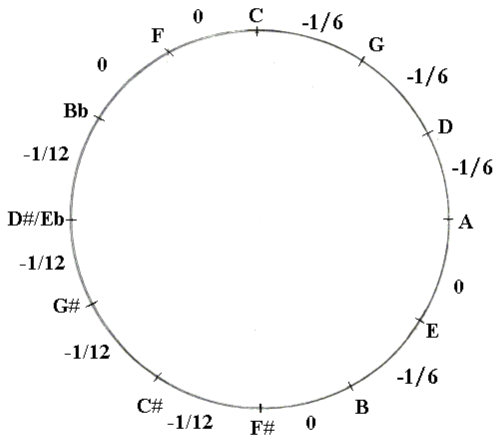

Figure 50 - O'Donnell temperament

An analysis of the thirds of this temperament shows that three of the major thirds on F, C and G are a little purer than Equal temperament thirds, four are exactly equal-tempered, a further four are just a step wide of equal-tempered, leaving the third on Ab a further step wide.

This temperament accords with Carl Philipp Emanuel Bach's requirement that most of the fifths be narrowed,[184] with Kirnberger's statement that Bach had taight him to tune all the thirds sharp,[185] and with the obituary's observation that "in the tuning of

[184] BACH, Carl Philipp Emanuel: *Versuch über die whare Art das Clavier zu spielen..., op. cit.*, p. 37.

[185] MARPURG, Friedrich Wilhelm: *Versuch über die musikalische Temperatur, nebst einem Anhang über den Rameau- und Kirnbergerschen Grundbaß, und vier Tabellen.* Berlin: Barnes, 1776 & Breslau: Johann Friedrich Korn, 1776, 320 pages, p. 213.

harpsichords he tempered tham so purely and correctly that all tonalities sounded beautiful and pleasing."[186]

Some other temperaments in Bach's day are similar to O'Donnell's temperament:[187]

1. Sorge 1744 III (good for *Kammerton* harpsichords) is also composed of four fifths reduced by 1/6 of a comma (in the same positions as O'Donnell's temperament) and four reduced by 1/12 of a comma. Moreover, the size of thirds are similar.
2. Neidhardt 1724 I – Village is similarly composed.
3. Neidhardt 1732 - 3rd circle IV temperament is almost identical with the only difference the position of the note Bb, which Neidhardt has 1/6 of a comma higher than in O'Donnell's temperament.

Bradley Lehman (2006, 4)[188]

Bradley Lehman also answers to John O'Donnell's article and he set out several points in which he is in disagreement with:

1. John O'Donnell says that his temperament is similar to Neidhardt 1732 - 3rd circle IV (only lowering Bb), Neidhardt 1724 I – Village and also Sorge 1744 III (good for *Kammerton* harpsichords). Lehman does not agree since only a few of the major thirds of these temperaments have the same size as O'Donnell's temperament, although it is easy to convert from one to the other in practice by simply moving a few notes. Consequently O'Donnell's temperament takes on different character as the other similar.
2. O'Donnell finds a similarity between the "Das" area of the decorative scroll and the tablature "Dis" symbol but the small C is not important for him. According to Lehman, "why choose the D to be so important instead of the C?"
3. O'Donnell's scheme interprets identical-looking loops in the drawing as two different sizes and Lehman thinks that nothing in the drawing suggests that the three loops at the left should indicate any differently-sized fifths among themselves; or similarly, for five loops at the right.
4. Lehman does not agree that Bach made a layout arranged chromatically. It is more simply to provide a more direct and practical scheme sequenced by the way one actually tunes a harpsichord, as a series of chained fifths or fourths.

Nevertheless, as a response to O'Donnel's article, Lehman formulated another temperament arranged chromatically in the scroll and it differs from O'Donnells in several questions. The practical instructions to tune this temperament by ear, a little evaluation and some examples to test it have been given in other posting.[189]

As a musician playing *The Well-Tempered Clavier*, Lehman thinks that both of his own layouts give a stronger focus to the music than O'Donnell's temperament does. He personally doesn't like the way the resulting major thirds sound in O'Donnell's temperament.

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[186] MIZLER, Lorenz Christoph: *Musikalischer Bibliothek..., op. cit.*, Vol. 4, pp. 172-3.
[187] *Vid.* "Appendix 4: Description of some historical 'good' temperaments".
[188] LEHMAN, Bradley: "Bach's temperament, Occam's razor, and the Neidhardt factor"..., *op. cit.*
[189] *Vid.* Section "Bradley Lehman (2006, 1) [Lehman III]".

Miklós Spányi (2007)[190] *[Kirnberger II & Spányi]*

Miklós Spányi declares himself in favour of the use of Kirnberger II temperament in Johann Sebastian Bach's works for harpsichord. He justifies his option according to historical facts and according to their practical results. Despite of the theoretical simplicity of this temperament, Spányi considers that this very criticized temperament can be used with very successful results if a suitable method of tuning is used. The structure of Kirnberger II temperament is shown in the following figure:

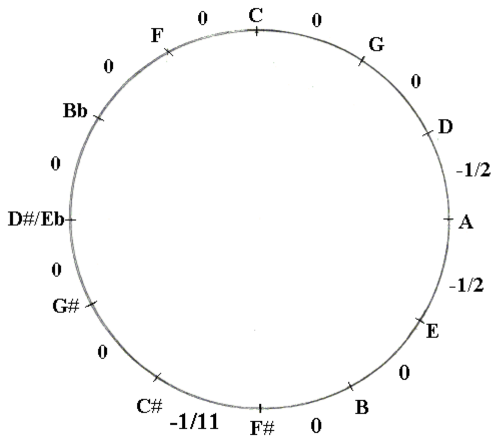

Figure 51 - Kirnberger II temperament

Spányi's article intends to clarify several ideas about Kirnberger's temperaments: On the one hand, the origin and the various possibilities of realizing Kirnberger's temperament and, on the other hand, the possibility that one of the Kirnberger's temperament was really Bach's own. The aim of Kirnberger's treatise was to lay down the principles of composition and music theory as thought and represented by J. S. Bach, through analysis and explanation of Bach's style and compositional techniques. In this very treatise, he gives short instructions about laying his called Kirnberger II temperament and he rejects the use of equal temperament.[191]

This temperament is unusually simple in relation to the rest of good temperaments of the 17th and 18th centuries but it follows the Kirnberger's principles about the characteristics of the ideal temperament:

- It should be easy to tune.
- It should not ruin the differences of key characters.
- It should present all intervals as much as possible in their melodic purity.[192]

This temperament has been very criticized. On the one hand, Friedrich Wilhelm Marpurg, who was one of Kirnberger's pupils, belonged to the group of musicians who tried to propagate Rameau's theories and, as a part of this, the use of equal temperament, against the ideas of his master. On the other hand, most of today's authors either completely neglects Kirnberger II temperament or mentions it as unusable and bad system.

Spányi thinks that the unsatisfactory results using this temperament are due to the lack of 'technique needed to lay the historical temperaments with the required

[190] SPÁNYI, Miklós: "Kirnberger's Temperament and its Use in Today's Musical Praxis". In: *Clavichord International*, Vol 11, No. 1 (May 2007), pp. 15-22. Dated 2006 at the end of the article.
[191] KIRNBERGER, Johann Philipp: *Die Kunst des reinen Satzes in der Musik..., op. cit.*, Part I, chapter 2, p. 13.
[192] KIRNBERGER, Johann Philipp: *Die Kunst des reinen Satzes in der Musik..., op. cit.*, Part I, chapter 1, p. 11.

tuning finesse.' The main problem is that a tuning must always be adapted to the requirements of different instruments and acoustics, aspects which are not taken into account making the temperament in their theoretically correct form. A further question is if historical tuners always set absolutely pure intervals when these were mentioned in tuning instructions. It has long been suspected that pure intervals were understood only as close-to-pure.

An enormous tolerance in the tuning of pure intervals can be observed working with organ pipes and stringed keyboard instruments. A pure third can be tuned considerably wider and a pure fifth considerably narrower than pure without hearing any beats.

Likewise, Pythagorean intonation is the usual basis for melodic intonation of strings instruments or singers and, because of this, Pythagorean thirds can be well used. Kirnberger remarks that keys with mostly Pythagorean thirds are more interesting than those having purer thirds,[193] then Pythagorean thirds can be accepted when situated in central keys.

Fifths can be tuned slightly less pure without their resulting beats are audible. These intervals are apparently pure but more colourful (or 'warmer'). This tolerance in the tuning of a chain of fifths lets to 'steal' a tiny amount of the comma for each one and the produced third becomes a little smaller than an 'official' Pythagorean third, but maintaining its fresh and radiant sound. If we reduced each fifth by half a cent (a theoretical amount just to demonstrate the tendency), the resulting third is 406 cents, instead of 408, a quite significant difference.

Moreover, this method of tuning gets a better result for the two fifths carrying the remains of the comma (D-A and A-E) since they become less narrow.

Before the explanation of his tuning instructions, Spányi gives some contemporary and later testimonies which can justify the use of Kirnberger's temperament:

- Carl Philipp Emanuel Bach[194] gives a short explanation of his preferred style of tuning: 'both sorts of instruments (the clavichord and the harpsichord) should be well tempered by reducing the utmost purity of most of the fifths so that the ear would hardly notice it'. By 'reducing the utmost purity' of the fifths he certainly meant a smaller amount of tempering than in ½ of a Pythagorean comma in equal temperament. 'Most of the fifths' (9?, 10?) can be reduced very slightly and the rest of the comma can be divided between the remaining (2?, 3?) fifths. The open question regards to which fifths are tuned narrower or wider.
- Barthold Fritz' instructions of tuning[195] don't specify any clear temperament but, certainly, they are very far from achieving anything like a real equal temperament. Perhaps he only intended to give general ideas about the technique of tuning and the maintenance of keyboard instruments. C. P. E. Bach congratulated him on his little booklet. About the fifths, he says that they would not have to be tuned too pure. He distinguishes between three grades of the pure fifth (none of the three beating):

[193] KIRNBERGER, Johann Philipp: *Die Kunst des reinen Satzes in der Musik..., op. cit.*, Part II, p. 72.

[194] BACH, Carl Philipp Emanuel: *Versuch über die whare Art, das Clavier zu spielen..., op. cit.*

[195] FRITZ, Barthold. *Anweisung, wie man Claviere, Clavecins, und Orgeln, nach einer mechanischen Art, in allen Zwölf Tönen gleich rein stimmen könne*. Leipzig, 1756, 1780, 1829. Dutch translation: *Onderwys, om Op eene Tuigwerkelyke wyze Clavieren, Clavecimbels, en Orgels, In alle 24 Toonen even zuiver te Stemmen, Op dat daar uit Zo wel de Major als Minor welluidend gespeeld kan worden*. Uit het Hoogduits vertaald door H. van Evervelt, en voorzien met eenige Aanmerkingen. Amsterdam: J.J. Hummel, 1757, 36 pages. Microfiche. Zug, Switzerland, Inter Documentation, 1976.

1. the ‘first purity’.
2. the ‘entirely pure’ (probably identical with C. P. E. Bach’s ‘utmost purity’).
3. the ‘superfluously pure’.

Fritz also advises to tune the fifths only to ‘first purity’ and never go as far as the ’superfluously pure’ grade. He doesn’t say anything about the place of the rest of the comma.

- In 1829, the south German organ-builder Ignaz Blasius Bruder[196] wrote down interesting and very important notes about organ building and tuning. It was finally the organ-builder Peter Vier[197] who recognized the obvious resemblance between Bruder’s and Kinrberger’s instructions. Indeed, both tuning schemes are very similar. About C-major and G-major (originally pure) triads, Bruder says: ‘tune the fifth only as much downwards [=reduced it] that you do not hear any more beats […] the major third amid them so sharp [=wide] that it is about to begin to beat’. About the other ‘pure’ fifth he says ‘all of these fifths should only have a little beat’.

Apart from these testimonies, it can be suggested that Kirnberger II temperament was in use for a surprisingly long time (even in the 19th century). It soon became popular and besides Germany it was introduced into France, England and Italy. In Germany, writings on music theory and/or organ-building mention it as late as about 1860-70.

Spányi’s have experienced with this temperament with both organs and stringed keyboards and he asserts that:

- The sound of the instruments in question becomes very energetic, vital and free, loud and rich in overtones.
- The temperament emphasizes the melodic lines so that they have extreme clarity and ‘purity’, and linearity is preferred to the harmony. This is in accordance with Kirnberger’s explanations about lineal intonation –and completely in contradiction to what is considered ‘authentic’ today.
- ‘Key characters’ are clearly outlined: the alternation of the extremities in chord structures – (mostly) pure on one and (mostly) Pythagorean triads on the other side as well as just a few triads between these extremes – F-major, D-major and A-major – give much more personality to the different tonalities than the (partly inaudible) subtleties among the many sizes of the major thirds in most other 18th century ‘well’ temperaments. While in other ‘well’ temperaments one often has the feeling that near keys with purer thirds are ‘better’ and remote ones with wider or Pythagorean thirds ‘worse’, in Kirnberger’s temperament one does not experience the key differences hierarchically. There are no ‘better’ or ‘worse’ keys, only very strongly contrasting ones.
- When the tuning has been laid well, the rather heavily tempered fifths D-A and A-E do not disturb, either, and despite of their relatively fast beats they positively contribute to the warm and colourful sound of the chords in which they occur.

Apart from his practical experiences and his historical arguments regarding to the use of the Kirnberger II temperament, another Spányi’s objective makes reference to

[196] *Vid.* BORMANN Karl: *Orgel- und Spieluhrenbau : Kommentierte Aufzeichnungen des Orgel- und Musikwerkmachers Ignaz Bruder (1829) und die Entwicklung der Walzenorgeln.* Series: Veröffentlichung der Gesellschaft der Orgelfreunde, 34. Zürich: Sanssouci-Verlag, 1968. ISBN 3-7254-0043-1.

[197] VIER, Peter: ‘Zur Frage der Rekonstruktion historischer Stimmungen’, *Acta Organologica*, Vol. 19 (1987), pp. 216-238.

a possible connection between this temperament and Bach's tuning praxis. It was Herbert Kelletat[198] who first established this possible connection.

There are not scientific musicological arguments to prove that Kirnberger's temperament corresponds with that of Bach's. Despite Kirnberger's treatise is declared as a summary of Bach's teachings, Kirnberger himself did not assert this at any moment.[199] Nevertheless, the connection could be conceivable since Kirnberger adored J. S. Bach, his music and his aesthetics. It seems rather unconvincing to suppose that Kirnberger would have published a temperament in his book which totally contradicted the tempering praxis of his master.

In conclusion, although Kirnberger II temperament is not originating from Bach, to perform Bach's music in this temperament is quite justified if it was the most famous unequal temperament known and used in much Europe in the century following its publication and it can still be considered as a product of the Bach-circle, appreciated and used by its central figures.

Moreover, other contemporary testimonies can support this affirmation. On the one hand, Kirnberger himself certainly tuned his clavichord according to his own temperament and, on teh other hand, Johann Friedrich Reichardt[200] listened to Kirnberger's performance of J. S. Bach's music on an 'extremely well tuned clavichord' with great admiration. Thus if Kirnberger found this temperament good enough to play the *Well-Tempered Clavier* in it, it should not hesitate to follow him.

Finally, Spányi gives a short guide to tuning 'Kirnberger II' in order to get a good result which is completely set out bellow:

Some preliminary remarks:

- As our very last step is to temper 'a', one should start laying the temperament from another note. I would suggest c1, but any other note within the chains of pure fifths is possible.
- Chains of pure fifths are realized as alternating fifths and fourths. By this one can avoid risky octave transpositions.
- When tuning a fifth upward (e.g. g0-d1) be careful to stop raising the higher note as soon as the beats have ceased being audible in order to stay on the somewhat narrow side of this interval.
- When tuning a fifth downward (e.g. f1-bb0) be careful to stop lowering the lower note as soon as the beats have ceased being audible in order to stay on the somewhat narrow side of this interval. If you approach the lower note (bb0) from bellow, after having tuned it pure try to raise it slightly but not as much to give beats.
- When tuning a fourth upwards (e.g. bb0-eb1), after having reached the pure interval try to raise the higher note as far as possible without noticing distinguishable beats in order to stay on the somewhat wide side of this interval.
- When tuning a fourth downward (e.g. f1-c1) try to lower the lower note as far as possible without noticing distinguishable beats in order to stay on the somewhat wide side of this interval. If you approach the lower note (c1) from bellow, stop raising it immediately as soon as beats have ceased being audible.
- My experience is that on most clavichords one hears the fifths best in the octave above middle C. On other instruments, however, using the octave bellow middle C suits better. Due to this I give two variants, A and B.

TUNING METHOD A:

198 KELLETAT, Herbert: *Zur Musikalischen Temperatur insbesondere bei Johann Sebastian Bach..., op. cit.*

199 Spányi cannot connect Kirnberger's temperament to the famous squiggles on the title page of the *Well-Tempered Clavier* either.

200 REICHARDT, Johann Friedrich: *Briefe eines aufmerksamen Reisenden die Musik betreffend.* Frankfurt & Leipzig, 1774 (Vol. 1); Frankfurt & Breslau 1776 (Vol. 2). Also available in: *International Music Score Library Project (IMSLP) Petrucci Music Library* [online]. In: <http://imslp.org/wiki/Briefe_eines_aufmerksamen_Reisenden_die_Musik_betreffend_(Reichardt,_Johann_Friedrich)>. [December 2010]. Quoted in KELLETAT, Herbert: *Zur Musikalischen Temperatur insbesondere bei Johann Sebastian Bach..., op. cit.,* 2nd edition (1981), p. 60; and KELLETAT, Herbert: *Zur musikalischen Temperatur, II. Wiener Klassik*. Kassel: Verlag Merseburger, 1982, 158 pages, p. 46.

- 1. Pure fifth: c1-g1
- 2. Pure fourth: g1-d1
- 3. Pure fifths and fourths: c1-f1-bb0-eb1-ab1-c#1
- 4. A close-to-pure major third: c1-e1

NB: This third, in the theoretical form of the temperament being pure, can be tuned somewhat larger than pure. I would let c1-e1 beat 2-3 times a second.

- 5. Pure fifth and fourth: e1-b0-f#1
- 6. Tempering of A: Tune a1 so that the fifth d1-a1 (narrow) beats more slowly than the fourth e1-a1 (wide). Their ratio is about 2:3 : in the same amount of time the fifth d1-a1 beats 2 times the fourth e1-a1 beats 3 times ('duplet-triplet' ratio).

TUNING METHOD B:

- 1. Pure fourth: g1-g0
- 2. Pure fifth: g0-d1
- 3. Pure fifths and fourths: c1-f0-bb0-eb0(or eb1)-ab0-c#1
- 4. A close-to-pure major third: g0-b0

NB: This third, in the theoretical form of the temperament being pure, can be tuned somewhat larger than pure. I would let g0-b0 beat about 1-2 times a second.

- 5. Pure fourths: b0-e1 and b0-f#0
- 6. Tempering of A: Tune a0 so that the fifth a0-e1 (narrow) beats more slowly than the fourth a0-d1. Their ratio is about 2:3 : in the same amount of time the fifth a0-e1 beats 2 times the fourth a0-d1 beats 3 times ('duplet-triplet' ratio).

NB: The fifth F#-C# results from these tuning steps and should not be tuned separately. In the original, theoretical form, this has the size of an equal-tempered fifth beating only very slowly. In our 'practical' version, f#0-c#1 can have about 1½-2 beats per second.

After this, all octaves are tuned, using the usual fifth-fourth interval checks. It is very important to tune octaves as correctly as possible to avoid an unnecessary stretching of the Pythagorean or close-to-Pythagorean thirds. It should also be ascertained by careful octave checks that the narrow fifths/wide fourths D-A and A-E do not become too narrow or wide in the upper octaves. To achieve a perfect result is not easy.

In the previous instructions, the narrowing of the fifths are expressed as approximated beat rates. Nevertheless, an exact mathematical proposition is given by Bradley Lehman.[201] Then, according to him, a layout for Spányi temperament can be represented in the following figure:

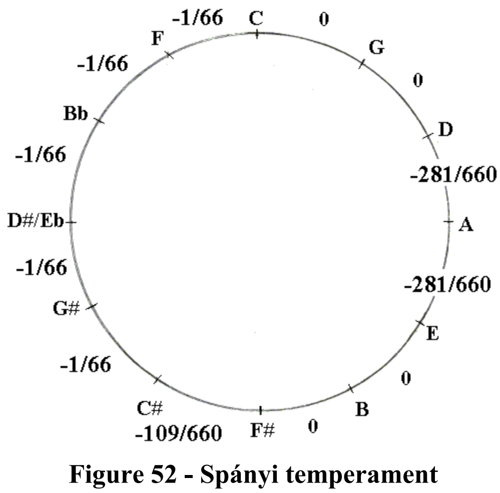

Figure 52 - Spányi temperament

Spányi do not give any connection between Kirnberger temperament and the famous squiggles on the tittle page of *The Well-Tempered Clavier*. As he says, he can not find it.

201 LEHMAN, Bradley: "Other 'Bach' temperaments"…, *op. cit.*

Graziano Interbartolo & Paolo Venturino (2007)[202] [Interbartolo-Venturino I, II & III]

Graziano Interbartolo & Paolo Venturino have designed three temperaments also based on Bach's drawing. Since June 2006 they have been doing some public performances with them, notably in a 1898 Bechstein piano.

All three temperaments have a very similar structure:

- Five consecutive fifths reduced by 1/4 of a comma between F and E.
- Four pure fifths between E and C# and between Bb and F.
- Three fifths wider than pure.

The five reduced fifths yields pure major thirds on F and C and also makes music sound very good in keys such as C major, F, major and D minor. The consequence is an excedent of 1/4 of a comma which has to be distributed in the three fifths which become wider than pure.

The difference among all three temperaments is the kind of comma (Pythagorean or *syntonic*) and the distribution of the amount of -1/11 of a *syntonic* comma:

- In temperament I, the comma is *syntonic* and this amount is distributed among the wider fifths giving intervals enlarged by +7/132 of a *syntonic* comma.[203]
- In temperament II, the comma is Pythagorean and the three wider fifths are enlarged by 1/12 of a comma.
- In temperament III, the comma is *syntonic* and the amount of 1/11 of a comma is placed between F# and C#, resulting only three pure fifths.

These temperaments are designed to be applied to the piano instead of the harpsichord. Thus it is logical that the result is not satisfactory enough for the harpsichord, according to Lehman's opinion. Lehman adds that "the Venturino/ Interbartolo temperament works better on piano than it does on harpsichord. (The extreme major 3rds sound less bright and problematic, since piano tone isweaker in vertones.) Their sounds samples are decent enough. It's sort of argueable that their temperament makes pianos sound "better", or at least more interesting, than equal temperament does."[204]

According to the authors, the system of tuning is based on the symbology in Bach's time and also on the tonal structure and form of the preludes and fugues which make up *The Well-Tempered Clavier*.

The layouts of the three temperaments given by Interbartolo and Venturino are shown in the following figures:

[202] INTERBARTOLO, Graziano & VENTURINO, Paolo: *BACH 1722: Il temperamento de Dio: Le scoperte e i significati del "Wohltemperirte Clavier"*. Firenze: Edizioni Bolla. Finale Ligure, 2007. ISBN: 8033064290935. More information in: INTERBARTOLO, Graziano & VENTURINO, Paolo: *BACH 1722 – Il temperamento de Dio* [online]. In: <http://www.bach1722.com>. [June 2010]. This temperament was deposited in the Tribunale di Savona. Repertorio 42.415. Raccolta numero 21.257. 27 June 2006.

[203] The detail of the operation is: (1/4-1/11)/3 = 7/132.

[204] *Vid.* LEHMAN, Bradley: "Other "Bach" temperaments"..., *op. cit.*, section "Interbartolo".

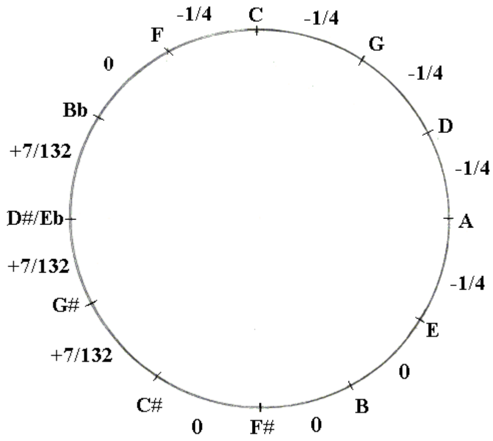

Figure 53 - Interbartolo - Venturino I temperament

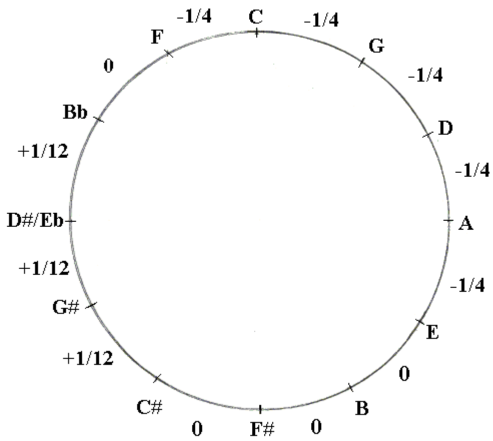

Figure 54 - Interbartolo - Venturino II temperament

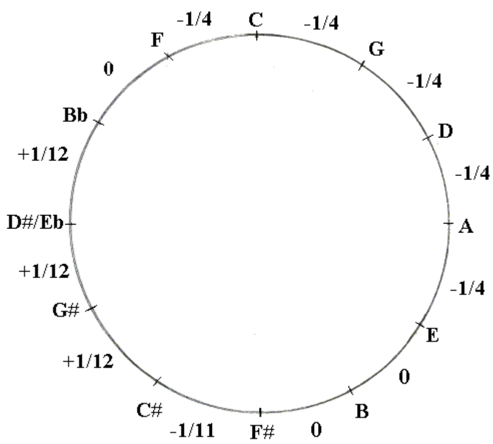

Figure 55 - Interbartolo - Venturino III temperament

Peter F. Williams (2007)[205]

Peter Williams starts his book making reference to Bach's obituary: “He knew how to give harpsichords so pure and correct a temperament in their tuning that all keys sounded beautiful and pleasing.”[206]

The first question is “what exactly the temperament was the *Obituary* was referring to, how long Bach had desired or practiced any particular form of it, and how it changed (as it surely had) during his lifetime.”

Regarding to *The Well-Tempered Clavier*, Williams asks himself for several questions:

1) If “well-tempered” means or not “equal temperament”. Three common views are: that Bach intended equal temperament; that he did not intend it; and that whichever this was, he wrote *The Well-Tempered Clavier* as indication of it. Although in German theory 'well-tempered' was not identical to 'equal tempered', by the 1720s it could have implied this in the context of a set of pieces in all the keys, assuming that 100% equality is practical. On the other hand, ‘well-tempered’ could mean a tuning system in which keys are all tolerable, but different and distinctive. Many writers since have reproduced such systems, arguing that in an unequal temperament Bach allows for the less sweet keys by tactfully underplaying any awkward harmonies. But since (i) some pieces were transposed for the collection and (ii) notation does not necessarily indicate how sustained the harmonies are, neither argument is reliable.

2) If a single tuning was intended for the complete work, if its very title is evidence for this and if each book is a set or cycle of pieces to be played as such in the given order. Although the pieces of the book are ordered rising by semitones, some interpreters change the order, for instance by dominants. The intention of the author could have been to tune the instrument for each piece / key as it was studied. Moreover, any of the pieces of the work modulates very far. This means that no key needs to be tuned except for the piece concerned.

3) If really the temperament was of vital importance to J. S. Bach. It is also probable that, like any composer, he was more interested in the differences between major and minor keys. The title-page of the *Book I* carefully specifies that all the major and minor keys are present.

Williams also makes reference to recent theories with regard to the diagram included in the autograph title-page. This line can indicate a temperament or a way to tune the clavier but, on the contrary, it might be a decoration or even a suggestion for an engraver. In relation to his question, Williams exposes several points in favor of this hypothesis and also against it.

Points in favor of this hypothesis are:

1) The diagram, unique on a Bach title-page, is otherwise puzzling.
2) The size of each curlicue varies as tempered intervals do.
3) Dividing the comma in this way was familiar from Werckmeister and Neidhardt.
4) Instrument-makers relied on wordless lines and yardsticks; why not a composer?

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205 WILLIAMS, Peter F.: *J. S. Bach: A Life in Music.* Cambridge: Cambridge University Press, 2007, 405 pp. ISBN 0-521-87074-7. Excerpt available in: “Tuning and temperament (An excerpt from Peter Williams’ *J.S. Bach: A Life in Music*, Cambridge University Press, 2007, pp. 333-338)”. In: *Bach Cantantas Website* [online]. 27 December 2007. <http://www.bach-cantatas.com/Articles/WilliamsTuning.pdf>. [December 2010].

206 MIZLER, Lorenz Christoph: *Musikalischer Bibliothek*, 4 Vols. Leipzig, 1754.

Points against the hypothesis are:

1) Five slighter curlicues appear as letter-ornaments on the title-page of the first *Anna Magdalena Book*, also dated 1722, and almost certainly in her hand. (In fact, did she, not the composer, add the curlicues to the first book of *The Well-Tempered Clavier*?).
2) A line of similar curlicues appears on each title-page of F. Suppig's treatises *Labyrinthus musicus* and *Calculus musicus* of 1722, with no apparent significance beyond (possibly) expressing the circularity of keys.
3) The diagram has to be viewed upside down, but the user is not told this.
4) If a small curving line looking like 'C' does indicate where the note C falls in the series of curlicues, it has to be read the right way up. It could be also a flourish on the letter 'C' of '*Clavier*' immediately below, like the 'C' of '*Concerto*' heading the *First Brandenburg Concerto* in the autograph fair-copy score.
5) No other instance is known in copies of *The Well-Tempered Clavier*.
6) One early copy (B. C. Kayser, a pupil) has a line with fewer curlicues but Kayser was not alerted to any significance.
7) Whether this temperament is implied is hypothetical; others can be inferred.

Apart from these arguments, Williams also notes that both the curlicues and the four words *Das Wohltemperirte Clavier oder* look like additions made after the full title-page was written. The diagram, the words and the date might all result from afterthought. This observation makes to ask oneself if the book received its title only later as the composer worked further on it. Likewise, the title-page's 'P' for '*Praeludia and Fugen ...*' is written with a flourish. This means that it could be the first word of the title.

Bernhard Billeter (2008)[207] [Billeter IIa, IIb, IIIa & IIIb]

Bernhard Billeter gives some alternative results to the interpretation of Bach's diagram based on a modification of Kirnberger III and Lehman II temperaments.[208] According to Freidrich Wilhelm Marpurg,[209] he declares that Bach tuned the major thirds slightly wider than pure. Nevertheless, it is not very clear if all thirds were tempered alike or they were different. Regardless of this question, Billeter thinks that it is possible to play all tonalities with an irregular good temperament and, moreover, the tonalities with a few accidentals sounds purer than those with more accidentals.

With regard to the scroll of Bach's autograph manuscript, Billeter thinks that it only gives information about the weighting in order to temper the fifths o the circle. So the first 5 fifths must be reduced, the following 3 fifths are pure and the last 3 fifths have to be also reduced by less than the first.

Bach's pupil Johann Philipp Kirnberger designed some temperaments which were possible to be tuned by ear. His third temperament (Kirnberger III) were more subtle than the other two. According to Billeter, these instructions have not to be understood mathematically and he suggests two modifications which result to be smoother than the original. The resulting temperaments can also be easily tuned by ear.

According to Marpurg, the third C-E has to be tempered slightly larger than pure. Billeter proposes an amount of 4 cents, that is, reducing each tempered fifth by 1

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[207] BILLETER, Bernhard: "Zur "Wohltemperirten" Stimmung von Johann Sebastian Bach: Wie hat Bach seine Cembali gestimmt?". In: *Ars Organi*, Vol. 56/1 (March 2008), pp. 18-21.

[208] *Vid.* Section "Bradley Lehman (2005, 1) [Lehman I & II]".

[209] MARPURG, Friedrich Wilhelm: *Versuch ubre die musikalische Temperatur..., op. cit.*

cent less than in the original temperament. Thus the fifths C-G-D-A-E result to be reduced by 4.5 cents whereas they are reduced by 5.5 cents (=1/4 of a *syntonic* comma) in the original version. The *schisma* is placed in the same interval than in the original temperament, that is F#-C#, and the extra 4 cents are placed in the fifth B-F#, following the Werckmeister example. This modification can be identified as "smooth" Kirnberger III or Billeter IIa temperament and its structure is given in the following table, where the amounts of tempering are expressed by *schismata*:

C#	0	G#	0	D#	0	A#	0	(F)
5.75		8		8		10		
A	2.25	E	0	B	2	F#	1	(C#)
4.25		2		4.25		4.5		
F	0	C	2.25	G	2.25	D	2.25	(A)
11		11		8.75		6.5		
Db	0	Ab	0	Eb	0	Bb	0	(F)

Figure 56 - Billeter IIa temperament – "Smooth" Kirnberger III temperament

The inconvenient of the previous solution is that the Pythagorean thirds on Ab-C and Db-F are very wide (10 *schismata* larger than pure). Because of this, Billeter proposes another modification of Kirnberger III temperament. If the *schisma* is placed in the fifth Eb-Bb instead of F#-A#, these thirds are slightly reduced (exactly by 2 cents). At this moment, 22 cents are left to be distributed among the rest of the circle. According to the author, the most favourable option is that they are distributed among the following five fifths: C-G-D-A-E (the fifths which are already tempered in Kirnberger III temperament) and B-F# (as it was made in the previous modification). If only integer numbers are used, 3 fifths have to be reduced by 4 cents (B-F#, D-A and C-G) and the other 2 by 5 cents (G-D and D-A). This solution replaces to the previous so-called "Bach-tuning" suggested by the same author in 1979.[210] It can be identified as "modified" Kirnberger III or Billeter IIb and its structure can be shown in the following table, where the amounts of tempering are expressed by *schismata*:

C#	0	G#	0	D#	1	A#	0	(F)
6.5		9		9		10		
A	2.5	E	0	B	2	F#	0	(C#)
4.5		2		4		4.5		
F	0	C	2	G	2.5	D	2	(A)
10		10		8		6.5		
Db	0	Ab	0	Eb	1	Bb	0	(F)

Figure 57 - Billeter IIb temperament – "Modified" Kirnberger III temperament

The structure of the original version of Kirnberger III temperament is shown in the following table, where the amounts of tempering are also expressed by *schismata*:

C#	0	G#	0	D#	0	A#	0	(F)
7.25		10		10		10		
A	2.75	E	0	B	0	F#	1	(C#)
2.75		0		2.75		5.5		
F	0	C	2.75	G	2.75	D	2.75	(A)
11		11		8.25		5.5		
Db	0	Ab	0	Eb	0	Bb	0	(F)

Figure 58 – Kirnberger III temperament (*schismata*)

[210] *Vid.* Section "Bernhard Billeter (1977) [Billeter I]".

Billeter disagrees with Lehman's solution because of these reasons:

- The amounts of 1/6 and 1/12 of a Pythagorean comma were used later and only by theoreticians such as Neidhardt and Vallotti. It is very difficult to tune these intervals by ear without the help of a monochord.

- The fifth Bb-F, which closes the circle, has to be tempered wider than pure, exactly by 1/12 of a Pythagorean comma. There are no instructions about tuning wide fifths in Bach's environment and period; even there are some warnings amongst them. These intervals could be found in some French temperaments due to the misunderstanding of Marin Mersenne's treatise.[211]

The importance of Lehman's discovery lies, more than in the very tuning instructions, in the demonstration of the use of an irregular good temperament instead of Equal Temperament in Bach's works.[212]

There are no too wide thirds in Lehman's temperament but the most impure third is placed on the note E instead of the farthest tonalities. Lehman tries to justify this fact making reference to *Cornet-ton* and *Cammerton* pitches. If the harpsichord played with string and wind instruments (*Kammerton* pitch), it would have to be transposed one tone lower (*Chornet-ton* pitch). Thus the third E-G# would correspond to the the interval F#-A#.

In order to solve this problem, Billeter suggests a modification of Lehman II temperament which can be identified as Billeter IIIa temperament. Its structure can be shown in the following table, where the amounts of tempering are expressed by *schismata*:

C#	1	G#	0.5	D#	0.5	A#	0	(F)
8.75		10		8		9		
A	2.25	E	0	B	0	F#	0	(C#)
3.25		2.25		4.75		6.75		
F	1.75	C	2	G	2	D	2	(A)
9		8.25		6.75		5.25		
Db	1	Ab	0.5	Eb	0.5	Bb	0	(F)

Figure 59 - Billeter IIIa temperament – First modification of Lehman II temperament

Other problem of Lehman's temperament is a preference for tonalities with flats. According to Emile Jobin,[213] the ornament placed on the left side of the scroll makes reference to the note F. This observation results in a displacement of Bradley's temperament by one fifth clockwise. The resulting solution is a second modified Lehman II temperament where the most reduced fifths start on C instead of F. This is a more favourable and practical solution which can be identified as Billeter IIIb temperament. Its structure is given in the following table, where the amounts of tempering are also expressed by *schismata*:

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[211] MERSENNE, Marin (pseud. Sieur de Sermes): *Traité de l'harmonie universelle, contenant la théorie et la pratique de la musique. Où il est traité des Consonances, des Dissonances, des Genres, des Modes, de la Composition, de la Voix, des Chants, & de toutes sortes d'Instruments Harmoniques.* Paris: Guillaume Baudry, 1627, 487 pages. Reedition: Paris: Sébastien Cramoisy, 1636/1637. Facsimile reprint by François Lesure (ed.): Paris: Éditions du Centre National de la Recherche Scientifique, 1963, 3 vols. English translation by Roger Edington Chapman: *Harmonie Universelle: The Books on Instruments.* The Hague: Martinus Nijhoff, 1957, 596 pages.

[212] Although it really belongs to Andreas Sparschuh, Michael Zapf and Keith Briggs. *Vid.* previous sections on these authors.

[213] JOBIN, Emile: "Bach et le Clavier bien Tempéré: Un autre éclairage à la découverte de Bradley Lehmann." In: *Clavecin en France* [online]. 30 September 2008. In: <http://www.clavecin-en-france.org>. [June 2010]. *Vid.* Section "Emile Jobin (2008) [Jobin]".

C#	0	G#	1	D#	0.5	A#	0.5	(F)
6.75		8.75		10		9.5		
A	2	E	2.25	B	0	F#	0	(C#)
5.25		3.25		2.75		4.75		
F	0	C	1.75	G	2	D	2	(A)
9		9		8.25		6.75		
Db	0	Ab	1	Eb	0.5	Bb	0.5	(F)

Figure 60 - Billeter IIIb temperament – Second modification of Lehman II temperament

In this second modified Lehman II temperament, the third E-G# is also reduced and the question about the tonalities with flats is also solved. The structure of the original Lehman II temperament is given in the following table, where the amounts of tempering are also expressed by *schismata*:

C#	1	G#	1	D#	1	A#	-1	(F)
9		10		9		8		
A	2	E	0	B	0	F#	0	(C#)
3		3		5		7		
F	2	C	2	G	2	D	2	(A)
9		8		7		6		
Db	1	Ab	1	Eb	1	Bb	-1	(F)

Figure 61 – Lehman II temperament (*schismata*)

Emile Jobin (2008)[214] *[Jobin]*

Emile Jobin, also as an answer to Bradley Lehman's article,[215] writes another article with regard the interpretation of the decorative scroll included in the manuscript of *The Well-Tempered Clavier*. Jobin agrees with the hypothesis that the the diagram makes reference to the tuning of the harpsichord but he interprets it in a different way.

Jobin pays attention to both elements situated in the extremes of the scroll. On the left side, the picture seems an F which represents the note F. The organ constructors used the same symbol to represent the same note below organ pipes in Bach's period. Likewise, on the right side, the picture seems a C which refers to the note C. Moreover, the small circle beside the letter C makes reference to a pure interval and the last symbol refers to the number 3. This means that the third C-E is pure and the interpretation of the symbol also indicates that the *syntonic* comma is the unit of measurement. It is not a coincidence that the reference notes given by Bach are F and C since these notes are also the pitch references in Bach's period.

As well as these symbols situated in the extremes of the scroll, the last loop is preceded by other letter C which also makes reference to the C note. Because of this, the last loop represents the fifth C-G whereas the extremes of the scroll represent the fifth F-C.

In the first capital letter of the title, the bar of the "D" is preceded by the symbol "Eb". Undoubtedly, this symbol refers to the note Eb and, consequently, the corresponding loop refers to the fifth G#-D#/Eb. It is not a coincidence that the eighth prelude of *The Well-Tempered Clavier* is written in Eb minor whereas the respective fugue is written in D# minor. Moreover, both points included in the symbol Eb indicate

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[214] JOBIN, Emile: "Bach et le Clavier bien Tempéré: Un autre éclairage à la découverte de Bradley Lehmann." In: *Clavecin en France* [online]. 30 September 2008. In: <http://www.clavecin-en-france.org>. [June 2010].

[215] LEHMAN, Bradley. "Bach's extraordinary temperament...", *op. cit.*

that the amount by which these fifths are tempered (and also the respective thirds) has to be negotiated.

Apart from this, it can be also considered that each kind of loop can be assigned to an unique kind of fifth.

According to these principles, Jobin's temperament can be described as follows:

1. The five loops on the right side are assigned to the fifths between C and B and they are reduced by 1/4 of a *syntonic* comma. Thereby two pure major thirds are obtained in C-E and G-B.
2. The three simple loops in the middle are assigned to the fifths between B and G# and they are pure. A tolerable major third is obtained in E-G#.
3. The extreme of the diagram is assigned to the fifth F-C and it is pure, according to the pitched in Bach's period.
4. The three loops on the left side are assigned to the remaining fifths between G# and F. The excedent yielded in this tuning has to be distributed among these fifths which result to be widener than pure and tempered by +7/132 of a *syntonic* comma (approximately +1/18 of a *syntonic* comma).

The excedent is calculated in this way:[216]

$$-\frac{5}{4}+3x=-1-\frac{1}{11}$$

$$x=+\frac{7}{132}\approx+\frac{1}{18}$$

and the resulting layout is shown in the following figure:

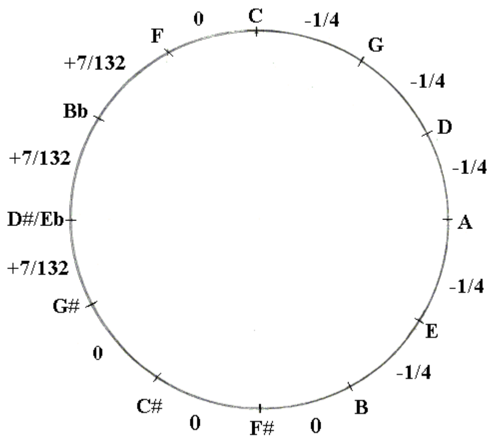


Figure 62 - Jobin temperament

[216] *Vid.* “Appendix 2: Definition of ‘good’ temperaments”, section “‘Good’ temperaments based on meantone temperaments”.

Claudio di Veroli (2009, 1)[217] *[Barnes]*

Claudio di Veroli set out a little survey about the temperament in Johann Sebastian Bach's *Well-Tempered Clavier* where he gives the reasons to rule out any temperament by Neidhardt, Marpurg, Kirnberger, Kelletat and Lehman. The remaining acceptable alternatives are Werckmeister III, Vallotti, Kellner and Barnes.

Regarding Barnes temperament, Di Veroli says that it is actually a slight variant of Vallotti, whereby B is tuned pure to E rather than to F#. Thus the very good major third G-B is tuned slightly worse in order to improve B-D#. Only 2 major thirds are now Pythagorean whereas all the 10 others are much better.

Di Veroli proposes and applies an improvement of the statistical method set out by Barnes[218] for *The Well-Tempered Clavier*.[219] For every temperament, the relevance of the 12 major thirds is now plotted against their mistuning, as it was done by Barnes. The improvement consisted basically in the application of a non-linear inverse (hyperbolic) regression curve instead of a linear regression. After this, two parameters were defined and calculated in order to evaluate the systems, both expressed in cents:

- Mean Absolute Deviation (MAD) of every major third from the ideal fitted curve.
- The absolute value of the Worst Fifth Deviation (WFD) for every interval of fifth.

The results for the four above-mentioned systems are as follows:

RANKING	TEMPERAMENT	MAD	WFD
1st	Barnes	2.376	3.910
2nd	Vallotti	2.959	3.910
3rd	Kellner	3.735	4.692
4th	Werckmeister III	3.727	5.865

The ranking above follows the MAD except for the last two, where the MAD is almost identical but the WFD is very different, hence the WFD is followed instead. It can be noticed that some temperaments, which superficially look relatively similar, really show significant differences after the evaluation.

With regard to the purity of the thirds in Bach's *Well-Tempered Clavier*, the best result is yielded for Barnes temperament. The result is also very similar for Vallotti temperament and the latter is better than either Kellner or Werckmeister III. Regarding to the fifths, the result is identical for both Barnes and Vallotti temperaments and poorer for Kellner and Werckmeister III.

[217] DI VEROLI, Claudio: *Unequal Temperaments. Theory, History and Practice. Scales, tuning and intonation in musical performance.* First edition: Bray (Ireland): Bray Baroque, November 2008. Second revised edition: Bray (Ireland): Bray Baroque, April 2009, 456 pages. eBook available in: *Soluciones de auto publicación e impresión de libros - Libros, eBooks, álbumes de fotos y calendarios en Lulu.com* [online]. In: <http://www.lulu.com/product/ebook/unequal-temperaments-theory-history-and-practice/4741955>. [January 2011]. Previous edition: *Unequal Temperaments and their Role in the Performance of Early Music. Historical and theoretical analysis, new tuning and fretting methods.* Buenos Aires (Argentina): Artes Gráficas Farro, 1978, 326 pages. More information in: DI VEROLI, Claudio: *Welcome to UNEQUAL TEMPERAMENTS!* [online]. In: <http://temper.braybaroque.ie/>. [January 2011].

[218] *Vid.* BARNES, John: "Bach's keyboard temperament...", *op. cit.*; and section "John Barnes (1979, 1) [Werckmeister III]".

[219] This study appeared in 1981, according to the information given by the author. Nevertheless, in his 1981 publication, where he also makes reference to Barnes's research, he concludes in a different way. *Vid.* Section "Claudio di Veroli (1980) [Tartini - Vallotti]".

The conclusion is that Barnes temperament is the most suitable option for *The Well-Tempered Clavier* by J. S. Bach. His layout is shown in the following figure:

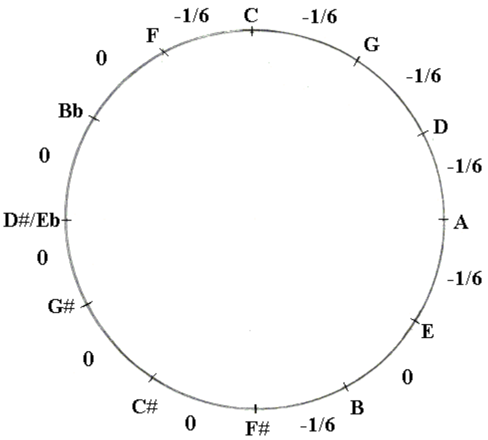


Figure 63 - Barnes temperament

Claudio di Veroli (2009, 2)[220] [Barnes, Di Veroli I & II]

After suggesting Barnes temperament according to the statistical method set out by Barnes himself, Claudio di Veroli intended to find out whether a yet unknown system could be found which "match" or fits Johann Sebastian Bach's *Well-Tempered Clavier* even better than Barnes temperament does. As well as, a secondary goal was to check for any significant difference between the two books of *The Well-Tempered Clavier*, pointing to a possible change in Bach's preferences over 20 years. For this objective he devised a computer program for the calculation. The procedure consists basically of these steps:

1. Enumerating all the possible 12-note partitions of the octave obtaining an output file with all possible temperaments.
2. For every temperament, checking the statistical fitting of the major thirds for each degree of the scale against the statistics compiled by Barnes.
3. Visually scrutinising the fitting of the curve for a group of selected temperaments, searching for peculiarities that may escape the statistical calculations.
4. Repeating the last two steps separately for both books of *The Well-Tempered Clavier*.

In order to bring this project down to a manageable size, the search has been initially restricted to those temperaments that only have some kinds of fifths. Three main cases were considered in the procedure, using two, three or four sizes of tempered fifths.

Some previous mathematical and statistical definitions are:

- Inverse Prominence Curve: a constant factor is applied to the reciprocals of the prominence values given by Barnes[221] in order to add up to a similar value in cents for both curves. This yields an ideal major-thirds deviation curve.

[220] DI VEROLI, Claudio: *Unequal Temperaments..., op. cit.*

[221] *Vid.* BARNES, John: "Bach's keyboard temperament...", *op. cit.*, p. 243, table 4; and section "John Barnes (1979, 1) [Werckmeister III]".

- Residual Sum (RS): sum of the absolute values of the 12 differences between the deviations of the major thirds on each degree of the scale respect Equal Temperament and the values of the Inverse Prominence Curve.

- Residual Sum of Squares (RSS): sum of squares of the same 12 differences.

Case 1: 2 sizes of fifths or "1/6 PC temperament search"

The first possibility in the searching for a Bach temperament is to select a restricted group of temperaments containing fifths either pure or flattened by 1/6 of a Pythagorean comma. This implies that these temperaments consist of 6 pure fifths and 6 flat fifths. Likewise, this group includes some of the main historical temperaments.[222]

If every fifth has only two possible sizes, the output consists of 924 possible circles of fifths. This was a manageable amount which can be loaded into a spreadsheet in order to evaluate their circles of major thirds according to Barnes's statistics. Two methods of evaluations can be considered:

1. Linear regression against inverse prominence:

The major thirds are sorted by the prominence values given by Barnes. Then, for each one of the 924 temperaments, linear regression is applied to its major third deviations against the reciprocals of the prominence. The RSS is found to be scarcely significant here and the only good criterion is to have a regression slope. A preliminary analysis rules out slopes smaller than 0.4. This left only 47 temperaments to evaluate, for which no regression coefficient is found to be meaningful. Because of this, the curve of each temperament is visually scrutinised against the inverse prominence to discard those that did not follow the expected shape. Barnes is the temperament whose curve fits better the prominence and Vallotti is the second best.

2. Sum of residuals against an inverse prominence curve:

For each one of the 924 temperaments, the discrepancy is scrutinised between its major thirds deviation curve and the Inverse Prominence Curve. The RS is found here slightly more meaningful than the RSS. The minimum values for RS are also given for Barnes temperament, which is found to be the optimal fit (RS=31). Vallotti, as a second best, is in 26th place, but still excellent (RS=35).[223] This is a confirmation of the optimally of Barnes and Vallotti for *The Well-Tempered Clavier*.

In order to find any significant differences between the two books of *The Well-Tempered Clavier*, both methods are also applied separately for each book. Two modified Barnes temperaments are found as alternatives for each book of the work but, in the authors opinion, the very small discrepancies found between them do not justify the use of these variants instead of Barnes's original temperament. Moreover, apart from being the best option for the whole work, Barnes temperament is suitable for all the other works by J. S. Bach and also an ideal general-purpose Good temperament for High Baroque music.

"1/12 PC temperament search" - Case 2: 3 sizes of fifths

The second possibility is to carry out the search using 3 sizes of fifths:

1. Pure fifths.
2. Tempered fifths reduced by 1/6 of a Pythagorean comma, that is, approximately 2 cents.
3. Tempered fifths reduced by 1/12 of a Pythagorean comma, that is, approximately 4 cents.

[222] Specifically, Di Veroli makes reference to the following temperaments: Vallotti, Vallotti/Young, Barnes, Marpurg's "F", and Neidhardt 1732 - 5th circle III.

[223] RS ranged from a minimum of 29 to a maximum of 118.

Thus the possible combinations are 3^{12}=531,441. Discarding those where the 12 fifths do not add up to the Pythagorean comma, only 73,789 are found to add up to the Pythagorean comma.

"1/12 PC temperament search" - Case 3: 4 sizes of fifths

The last possibility in the searching for a Bach temperament is to consider the group of all temperaments with the following four sizes of fifths:

1. Pure fifths.
2. Tempered fifths reduced by 1/6 of a Pythagorean comma, that is, approximately 2 cents.
3. Tempered fifths reduced by 1/12 of a Pythagorean comma, that is, approximately 4 cents.
4. Tempered fifths enlarged by 1/12 of a Pythagorean comma, that is, approximately 2 cents.

This is a much more complex computational task since there are 4^{12}=16,777,216 possible mathematical combinations. Discarding those where the 12 fifths do not add up to the Pythagorean comma, only 534,964 possibilities correspond to valid circular temperaments.

Moreover, it has been taken into account that the use of one sharp fifth implies 2 pure fifths less in the circle, that is 4 significant cents, since the extra 2 cents have to be compensated. The use of 3 different sizes of tempered fifths also increases significantly the tuning difficulty. However, the advantage is that a more accurate fit to the Prominence curve is possible.

Before continuing the evaluation of both cases 2 and 2, some previous observations must be taken into account.

Some distortion has been found in the Prominence curves which is caused by the "non smooth" values over D and A, that is to say, the prominences for the intervals D-F# and A-C#. In order to obtain a smoother curve, Di Veroli proposes to obtain an amended curve which could be also evaluated for both cases 2 and 3. Two alternatives are considered:

1. To amend the curve on the interval A-C#, implying the hypothesis that Bach did use more frequently an interval not very consonant.
2. To amend the curve on the interval D-F#, implying that Bach did not use frequently enough a good interval.

Clearly the second alternative corresponds to a more likely hypothesis and is therefore selected. According to this, two more possibilities are considered in the search of the temperament: to use either the original Prominence curve, with D-F#=48, or an Amended curve with D-F#=68.

After discarding the invalid temperaments, the number of possible temperaments is still too much elevated. Because of this, a new improved computer program is devised in order to perform a preliminary evaluation obtaining an output file with only a few hundreds of temperaments. The results can then be imported into a spreadsheet for final analysis and visual check of the curves. This ranking consists to search for temperaments showing minimum discrepancy against the Inverse Prominence curve and this value can be calculated according to either RS or RSS.

At this moment, two methods of evaluation are applied to the obtained set of selected temperaments for both cases 2 and 3, serching either for the statistical best fit or for the visual bet fit.

The first method, which search for the statistical best fit, is run according to three sets of binary situations, that is, 8 in total:

- Using the original Prominences or the amended ones (abbreviated PO and PA).

- Using 3 or 4 sizes of fifths (abbreviated V3 and V4)
- Ranking by either RS or RSS.

A special worksheet is prepared for each run and, after this, for each one of the 8 worksheets, a first statistical method is followed in order to search a "statistical best fit":

- Sorting the temperaments by RS or RSS.
- The temperament with best (that is minimal) RS or RSS is properly labelled.
- Since most of the temperaments yield the same value for the RS/RSS (sometimes as many as 40 temperaments), the best fit has to be finally selected by visual examination of the circles of thirds and fifths.

The results from the 8 runs are summarised in the following table, where the units of definition are equivalent to a -1/12th of a Pythagorean comma:

Parameter	Eb	Bb	F	C	G	D	A	E	B	F	C#	G#	RS/RSS	Comment
PO,V3,RS	0	1	2	2	2	1	2	0	2	0	0	0	22.5	Very good
PO,V3,RSS	0	2	2	2	2	1	2	0	1	0	0	0	72.5	Poor
PO,V4,RS	0	2	2	2	1	2	2	0	1	1	-1	0	21.3	Very good
PO,V4,RSS	0	2	2	2	2	1	2	0	1	1	-1	0	54.3	Poor
PA,V3,RS	0	1	2	2	2	1	2	0	2	0	0	0	17.6	Same as PO,V3,RS
PA,V3,RSS	0	1	2	2	2	1	2	0	2	0	0	0	53.3	Same as PO,V3,RS
PA,V4,RS	0	2	2	2	1	2	2	0	1	1	-1	0	16.4	Same as PO,V4,RS
PA,V4,RSS	0	2	2	2	1	2	2	0	2	0	-1	0	34.7	Poor

It can be noticed that some of the results are identical and there are therefore only 5 different temperaments. After an individual examination, 3 of the results are found to be either a "poor" solution or a "poor visual fit". Finally only 2 temperaments are left, which are precisely the ones that have been found in repeated runs. Nevertheless, the two final solutions are not considered definitive. As it will be seen bellow, visual inspection of the whole list identifies an even better temperament, according to a second visual method of evaluation.

In the second method of evaluation, which search for the visual best fit, the temperaments selected are imported to the worksheet and visually scrutinised in order to look for a "visual best fit", regardless of the obtained values of the RS or RSS for each one. One temperament is selected as a visual fit for each of the 8 worksheets or possible situations. The differences between RS and RSS values do not have any influence in the visual selection. Thus a unique temperament is selected as the best "visual fit" for all the possible values of RS and RSS. This means that there would be not 8 but only 4 possible "visual best" temperaments. Moreover, since the PO and PA curves are different, they can also be evaluated regardless of whether PO or PA was selected as the ideal curve. Thus, also a unique temperament is selected as the best "visual best" and the possible alternatives are finally reduced to only 2. Summing up, only 2 ideal temperaments are obtained according to the difference between using 3 or 4 sizes of fifths. The table bellow shows the final best results with the RSS for the Amended Prominence. The units of definition of the temperaments are also expressed in -1/12th of a Pythagorean comma. Barnes temperament has been also added since it corresponds to the optimal solution for 2 fifth sizes:

V sizes	Eb	Bb	F	C	G	D	A	E	B	F	C#	G#	RSS	Temperament name
2	0	0	2	2	2	2	2	0	2	0	0	0	91.1	**Barnes's Bach**
3	0	1	2	2	2	1	2	1	1	0	0	0	53.6	**WTC Optimal**
4	0	1	2	2	2	1	2	1	1	0	-1	1	43.1	**WTC Optimal+**

The optimal solutions obtained can be identified as Di Veroli I and Di Veroli II temperaments and their layouts are shown in the following figures:

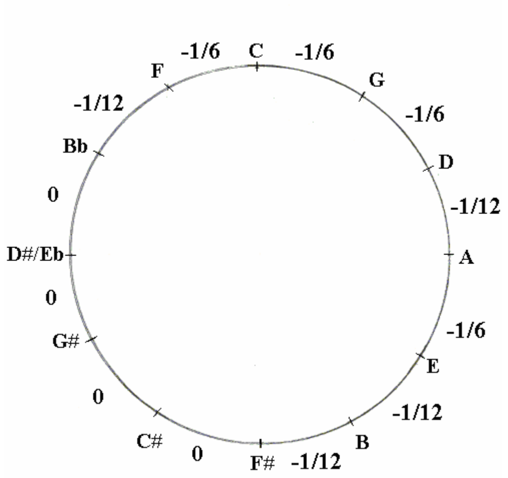


Figure 64 - Di Veroli I temperament - WTC Optimal

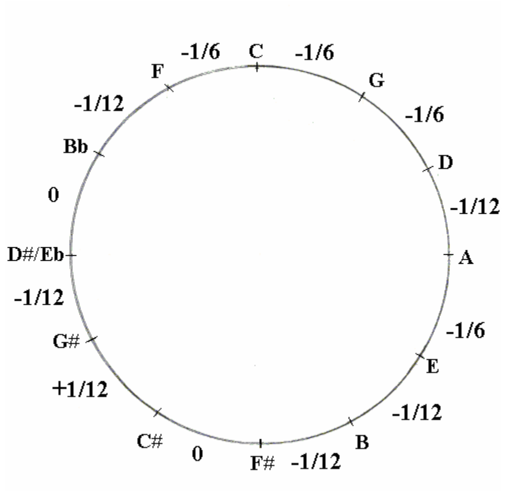


Figure 65 - Di Veroli II temperament - WTC Optimal+

As a conclusion, it can be noticed that with 3 fifths only, the RS/RSS are excellent predictors of a good final visual fit. Nevertheless, this is not true when 4 fifths are used since the best temperament (according to the visual fit) is slightly but significantly more deviated than the statistical best fit. The computer analysis and the visual scrutiny have also proved that no temperament with 4 sizes of fifths exists as a "perfect fit" for the prominence reciprocals in *The Well-Tempered Clavier*. The WTC Optimal+ curve clearly follows the Amended Prominence Curve slightly better than Barnes, especially in the major thirds involving accidentals. There is also a tendency of WTC Optimal+ to favour the flats more than Barnes. As well as, the 4 sizes of fifth in WTC Optimal+ increase significantly the tuning difficulty and implies "a radical departure from the simple Werckmeister-II-like Good-temperament tuning practices in

Bach's times and places.” No such a thing as an "ideal" or "rediscovered" temperament exists for *The Well-Tempered Clavier* by Johann Sebastian Bach.

John Charles Francis (2011)[224]

Francis noticed that the graphic included in the manuscript of *The Well-Tempered Clavier* shows three central loops with three single-knotted anticlockwise loops to the left and five doubleknotted clockwise loops to the right:

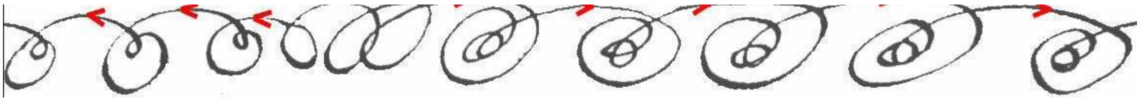
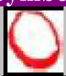
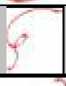


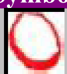
Figure 66 - Graphic included in the manuscript of *The Well-Tempered Clavier*

The central loops represent the tempering of fifths on the subdominant, tonic and dominant of C; the anticlockwise loops represent the tempering of fifths doing towards the flats; the clockwise loops represent the tempering of fifths going towards the sharps; and the end points of the graph represent enharmonic spellings of the interval closing the circle of fifths. This interpretation is summarized in the following table:

Symbol	Interpretation
	Tuning for subdominant, tonic and cominant fifths of C
	Tuning for fifths towards the flats
	Tuning for fifths towards the sharps
	Enharmonic spellings of interval closing the circle of fifths
	Enharmonic spellings of interval closing the circle of fifths

These topological characteristics of the symbols allow any arbitrary temperament but derive an unique solution.

The temperament resulted from his previous articles[225] has the following equivalence:

Symbol	Number of small loops	Tempering (Hz)
	0	0
	1	1
	2	2
	1	1
	2	2

[224] FRANCIS, John Charles: “Bach’s Well Tempered Tuning”. In: *Eunomios. An open online journal for theory, analysis and semiotics of music* [online]. 3 February 2011. In: <http://www.eunomios.org/contrib/francis6/francis6.pdf>. [March 2011]. Also available in: *Bach Cantantas Website* [online]. February 2011. <http://www.bach-cantatas.com/Articles/Bachs_Well_Tempered_Tuning.pdf>. [March 2011].

[225] FRANCIS, John Charles: “The Esoteric Keyboard Temperaments...”, *op. cit.* & RANCIS, John Charles: “Das Wohltemperirte Clavier...”, *op. cit.*

And this yields the *Cammerton* temperament 9-1, which is paired with its *Cornet-ton* transposition 7-2 (both described above).[226]

It easy to notice the equivalence of this information with the distribution of the fifths in the *Cammerton* temperament 9-1, that is: the fifths F-C-G-D don't beat; the fifths D-A-E-.B-F#-C# bets twice a second; the fifths Ab-Eb-Bb-F beats one a second; finally, the fifth C#-Ab beats one a second. The *Cornet-ton* temperament results from transposition of the previous scheme with the difference that the fifth placed at the end point beats twice a second. In the *Cornet-ton* temperament, Bb would have to be considered as the tonic.

Thomas Glueck[227] [Glueck]

Thomas Glueck proposes another temperament which is also an interpretation of the diagram included in the title page of the autograph score of *The Well-Tempered Clavier* by Johann Sebastian Bach. Its definition is given in the following figure:

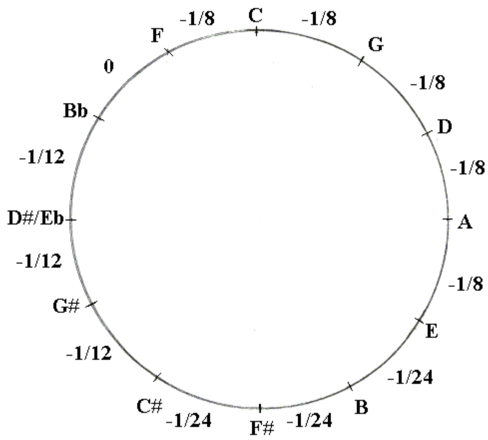


Figure 67 - Glueck temperament

The Bach's '1722 Seal'

Another graph which has been the subjet of several tuning interpretacions is the Bach's '1722 Seal':

[226] *Vid.* Section "John Charles Francis (2005, 1) [Francis II & III]".
[227] *Vid.* LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning"..., *op. cit.*, p. 620. Contact Thomas Glueck at glueck10@aon.at.

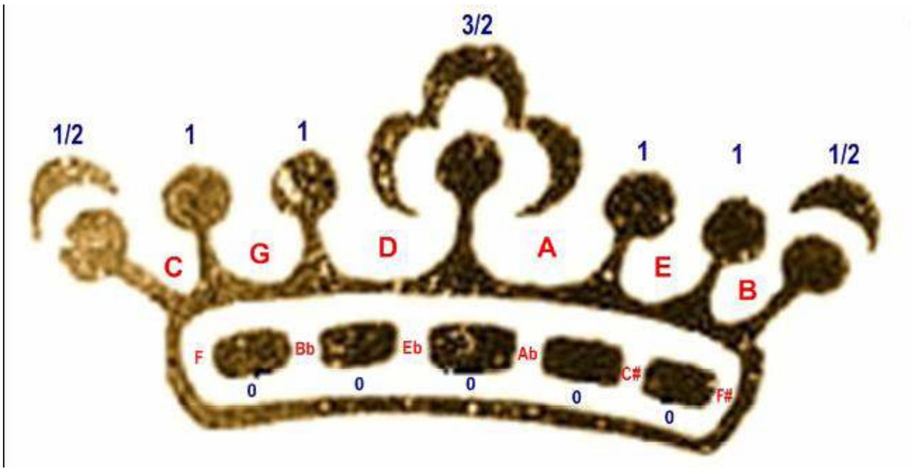


Figure 68 - Bach's '1722 Seal'

Several explanations have been given by authors such as Herbert Anton Kellner (the first person who gave an interpretation of the Bach's '1722 Seal'),[228] John Charles Francis[229] and Andreas Sparschuh.[230]

A more complete explanation of this question, which is in relation to the theme of this paper, is beyond my scope in this research.

Others

Other authors who have recently given more solutions to the question of the temperament in Bach's *Well-Tempered Clavier* are:

- Rudolf Rasch (1981).[231]
- Balint Dobozi (2000).[232]
- Bradley Lehman's response to Charles Francis (2005, 2) (2005-2006).[233]

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228 Several articles on this question were produced by Herbert Anton Kellner. A fairly complete list of his works is given in the section "Complementary bibliography (Kelletat and Kellner)".

229 FRANCIS, John Charles: "Tuning Interpretation of Bach's '1722 Seal' as Beats Per Secons". In: *Bach Cantantas Website* [online]. 17 April 2006. <http://www.bach-cantatas.com/Articles/Bach_Seal.pdf>. [June 2010]. Also available in: *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. 17 April 2006. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. File posted in: <http://launch.groups.yahoo.com/group/clavichord/files/>. [June 2010]. Also available in: *Yahoo! Bach_tunings group: Modern JSB tempering and more about JSB* [online]. 17 April 2006. Group bach_tunings@yahoogroups.com in <http://launch.groups.yahoo.com/group/bach_tunings/>. File posted in: <http://launch.groups.yahoo.com/group/bach_tunings/files/>. [June 2010]. 7 pages.

230 SPARSCHUH, Andreas: Several postings in: *Yahoo! Bach_tuning group: Welcome to the Alternate Tunings Mailing List* [online]. Group tuning@yahoogroups.com in <http://launch.groups.yahoo.com/group/tuning/>. [June 2010].

231 RASCH, Rudolf: "Wohltemperirt engelijkzwevend". In: *Mens en Melodie*, Vol. 36 (1981), p. 264; RASCH, Rudolf: "Does «Well-tempered» mean «Equal-tempered»?" In: WILLIAMS, Peter F. (ed.): *Bach, Handel, Scarlatti: Tercentenary Essays.* Cambridge: Cambridge University Press, 1985, pp. 293-310.

232 DOBOZI, Balint: "Vergleich verschiedener wohltemperierter Stimmungen anhand von J. S. Bachs Fuga in As-Dur BWV 886 aus dem zweiten Band des Wohltemperierten Klaviers und deren Frühversion, der Fughetta in F-Dur BWV 901." In: *fres.ch - since 2000. Musikwissenschaft* [onine]. October 2000. In: <http://www.fres.ch/bd/content/music/bach.html>. [January 2011].

233 LEHMAN, Bradley: "Further response to Charles Francis's third paper". In: *Johann Sebastian Bach's tuning* [online]. October 2005 & April 2006. In: < http://www-personal.umich.edu/~bpl/francis-paper-3.html> & <http://www.larips.com>. [December 2010]. Response to: FRANCIS, John Charles: "Das Wohltemperirte Clavier...", *op. cit.*

- Johan Broekaert (2007).[234]
- Andreas Sparschuh (2005-2008). [235]
- Bradley Lehman's response to Claudio di Veroli (2009).[236]
- Emmanuel Amiot (2009).[237]
- Claudio di Veroli's response to Bradley Lehman (2010).[238]

The explanation of their propositions is beyond my scope in this paper, as are other works by authors such as Anton Kellner and Herbert Anton Kelletat.[239]

More information on the Internet

More information can be obtained on the Internet. To begin, lots of the works that have been cited and commented on can be downloaded from the Internet. Indeed, some of them are only published online.

Apart from all the bibliographical information given in the previous paragraphs, more information and commentaries posted by several of the same authors of the previous articles (and others) can be found in a variety of forums or discussion groups on the Internet. Some of the sources considered in the previous survey are only avaliable via these discussion groups since they have not been published anywhere. The information included there is available by subscription and, in most cases, after being accepted by the moderator of the group. These are the titles of the groups:

Yahoo! Bach tunings.[240]
Yahoo! Clavichord.[241]
Yahoo! Tuning.[242]
Yahoo! Harpsichord.[243]

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[234] BROEKAERT, Johan: "Bach- and Well-Temperaments or Western Classical Music. A proposal for an objective musical definition. A herewith connected proposal for an objective evaluation model." In: *Welkom bij DeDS* [online]. 21 October 2007. In: <http://home.deds.nl/~broekaert/Welltempered_3_5_all_figures.pdf>. [January 2011].

[235] SPARSCHUH, Andreas: Several postings in: *Yahoo! Bach_tuning group: Welcome to the Alternate Tunings Mailing List* [online]. Group tuning@yahoogroups.com in <http://launch.groups.yahoo.com/group/tuning/>. [June 2010].

[236] LEHMAN, Beadley: "Unequal temperaments". In: *The Viola da Gamba Society Journal,* Vol. 3, Part 2 (2009), pp. 137-163. Also available in: *The Viola da Gamba Society* [online]. In: <http://www.vdgs.org.uk/files/VdGSJournal/Vol-03-2.pdf>. [January 2011].

[237] AMIOT, Emmanuel: "Discrete Fourier Transform and Bach's Good Temperament". In: *Music Thoery Online: A Journal of Criticism, Commentary, Research and Scholarship,* Vol. 15, No. 2 (June 2009) [online]. Available in <http://mto.societymusictheory.org/issues/mto.09.15.2/mto.09.15.2.amiot.html>. [January 2011].

[238] DI VEROLI, Claudio. "Unequal Temperaments: Revisited", In: *The Viola da Gamba Society Journal* Vol. 4 (2010), pp. 164-182. Also available in *The Viola da Gamba Society* [online]. In: <http://www.vdgs.org.uk/files/VdGSJournal/Vol-04.pdf>. [May 2011]. Also available in: DI VEROLI, Claudio: *Welcome to UNEQUAL TEMPERAMENTS!* [online]. In: <http://temper.braybaroque.ie/VdGS-Lehman-Rebuttal-Vol-04-CDV.pdf>. [June 2010].

[239] A complementary bibliography of Herbert Kelletat and Herbert Anton Kellner is given in the "Bibliography", section "Complementary bibliography (Kelletat and Kellner)".

[240] *Yahoo! Bach_tunings group: Modern JSB tempering and more about JSB* [online]. Group bach_tunings@yahoogroups.com in <http://launch.groups.yahoo.com/group/bach_tunings/>. [June 2010].

[241] *Yahoo! Clavichord group: A forum for makers, players and enthusiasts of the clavichord* [online]. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/clavichord/>. [June 2010].

[242] *Yahoo! Tuning group: Welcome to the Alternate Tunings Mailing List* [online]. Group clavichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/tuning/>. [June 2010].

Yahoo! Harpsichords.[244]

Yahoo! Bach keyboard.[245]

LISTSERV HPSCHD-L.[246]

Moreover, lots of personal (and other) sites on the Internet contain more information about temperament in Bach's *Well-Tempered Clavier* and about tuning and temperament generally as well as more bibliographical references on these themes.[247]

Bradley Lehman has has posted several of his own articles on his personal Web site,[248] including those published in paper, a summary of all information he has about temperament in Bach's *Well-Tempered Clavier* and some responses to articles regarding his own works. Here is the list of the main articles posted on his own Web site - *Johann Sebastian Bach's tunin*g:

- Johann Sebastian Bach's tuning (2005-06).
- Bach's Art of Temperament (April 2006, with several improvements added June 2006).
- The 'Bach temperament' and the clavichord.
- Practical temperament instructions by ear.
- "Ordinary" Temperament. "Extraordinary" Temperament.
- Carl Philipp Emanuel Bach.
- Further response to Charles Francis's third paper (October 2005 & April 2006).[249]
- Bach-style keyboard tuning (October and November 2006).[250]
- Bach's temperament, Occam's razor, and the Neidhardt factor (November 2006).[251]
- Other "Bach" temperaments.

In the section "Other 'Bach' temperaments", there are commentaries and information on the following authors: Herbert Kelletat, Herbert Anton Kellner, John Barnes, Bernhard Billeter, Mark Lindley, Andreas Sparschuh, Martin Jira, Michael Zapf, John Charles Francis, Daniel Jencka, Emile Jobin, Richard Maunder, Kenneth Mobbs, Alexander Mackenzie of Ord, George Lucktenberg, Graziano Interbartolo, Paolo Venturino, Ibo Ortgies, John O'Donnell, Miklós Spányi, Peter F. Williams and Claudio di Veroli.

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243 *Yahoo! Harpsichord group: A forum for builders, players and enthusiasts of the harpsichord and its music. The list is restricted* [online]. Group harpsichord@yahoogroups.com in <http://launch.groups.yahoo.com/group/harpsichord/>. [June 2010].

244 *Yahoo! Harpsichords group: A meeting place for lovers of the harpsichord* [online]. Group harpsichords@yahoogroups.com in <http://launch.groups.yahoo.com/group/harpsichords/>. [June 2010].

245 *Yahoo! Bach-keyboard group: Bach playd on piano, harpsichord, or clavichord* [online]. Group bach-keyboard@yahoogroups.com in <http://groups.yahoo.com/group/bach-keyboard/>. [June 2010].

246 *LISTSERV HPSCHD-L: Harpsichords and Related Topics* [online]. Group HPSCHD-L@LIST.UIOWA.EDU in <http://list.uiowa.edu/scripts/wa.exe?A0=HPSCHD-L>. [December 2010]. This LISTSERV server is located at LIST.UIOWA.EDU. The University of Iowa.

247 *Vid.* 'Bibliography' to obtain more information about these references on the Internet.

248 LEHMAN, Bradley: *Johann Sebastian Bach's tuning* [online]. In: <http://www-personal.umich.edu/~bpl/larips/> & <http://www.larips.com>. [December 2010]. Bradley Lehman's Web site.

249 Response to: FRANCIS, Charles: "Das Wohltemperirte Clavier...", *op. cit.*

250 Response to: LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning"…, *op. cit.*

251 Response to: O'DONNELL, John: "Bach's temperament, Occam's razor...", *op. cit.*

Some conclusions on the meaning and interpretation of the decorative scroll

All last solutions given to the problem of temperament in the *Well-Tempered Clavier* by Johann Sebastian Bach are based on the interpretation of the decorative scroll existing on the cover of the autograph of the first volume of his work. According to the summary given by Thomas Dent[252] the interpretation of the diagram is based on two immutable principles and other decisions to be taken to determine the temperament applied to the work.

The two immutable principles are:

a) There are three tempered fifths of one size and five tempered fifths of another size, and three pure fifths separating them. There is also a remainder fifth which determines how much these two classes of fifth are to be tempered.
b) One class of tempered fifth is twice as far as the other from being pure.

The decisions to be taken are:

a) The direction in which the diagram is to be read.
b) The starting note.
c) The type of tempered fifth to be tempered by 2 units and that by 1 unit.
d) The size of unit.

The most important of these decisions are the choice of *direction* and the choice of *which fifths are tempered more*. Having fixed these, the choice of unit is constrained so that the remaining fifth is not very wide or narrow; and the choice of starting note is more or less fixed by the expectation that F or C should have the purest major third.

These principles can be defined in a more generic way in order to consider all of the solutions given in al of the known articles.

In most cases, one class of tempered fifth is not twice as far as the other from being pure. With the exceptions of Lehman II & III, Ponsford II, Lindley-Ortgies II, Interbartolo-Venturino I, II & III & Jobin, all values of the tempering are negative.

In some cases, there are no pure fifths and, consequently, there are three classes of tempered fifths. In these cases, simple loops don't represent pure fifths but a very small value of tempering.

The remaining fifth is also determined as a function of the rest of the circle. In several cases, the resulting size is the same as one of the other kinds of fifths. In lots of cases it is also zero or even a positive value. In the unique case of Francis II & III, two kinds of temperament are determined for the same scroll depending on the value of this remaining fifth, which can take the value of one of the two possible values of tempering. This is precisely the reason for leaving an open fifth in the circle with two different signs drawn in every extreme of the scroll. These signs refer to each of the two possible cases.

With the exception of O'Donnell & Lehman III, each loop of the scroll represents one of the fifths of the circle following the normal order, that is reading it clockwise. For the cases given by O'Donnell & Lehman III, each loop represents one fifth, at same as the rest of cases, but the order of these fifths follows the chromatic scale. Likewise, these solutions don't follow the general principle according to which there are three tempered fifths of one size and five tempered fifths of another size, apart from the remaining fifth and the other three pure or not pure fifths.

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[252] *Vid.* DENT, Thomas: "Zapf, Lehman and other Clavier-Well-Temperaments"…, *op. cit.*

Three units have been considered in all interpretations given for the decorative scroll: fractions of the Pythagorean comma, fractions of the *syntonic* comma and beat rates. The majority of solutions are based on the *comma pitagorico*. Very few cases are based on the other two units. For the case where the syntonic comma has considered the unit of reference, a remaining fifth reduced by a *schisma* has to be added to the circle, apart from the remaining fifth represented by the extremes of the scroll. The *schisma* has been placed in different ways depending on the case. It is not placed necessarily in the same place of the fifth of the extrem. In several of these cases, the *schisma* has been distributed among three of the fifths giving values of tempering represented by more complicated fractions of the *syntonic* comma.

Of course, in all cases, the fractions of the comma have to fulfil the conditions given for 'good' temperaments. The specific case of Thomas Dent follows another principle which doesn't correspond to this condition. He approximates the *schisma* by a twelfth part of the *syntonic* comma resulting in a different value of the sum of all tempering values.

Thus we can redefine the principles and decisions given by Thomas Dent.

The immutable principles would be:

a) There are three tempered fifths of one size and five tempered fifths of another size, and three other fifths separating them which are pure in general but not always. In some cases these fifths are tempered by a very small value.
b) There is also a remainder fifth which determines how much these two classes of fifth are to be tempered. Nevertheless some methods define two temperaments according to two possibilities for the value of the 'remaining' fifth. This is only possible when the unit of measurement is the beat rate.

The decisions to be taken are:

a) The direction in which the diagram is to be read.
b) The starting note.
c) The order of the intervals, in the majority of cases, follows the order of the circle of fifths.
d) The size of the unit.
e) The value of the tempering units for each kind of fifth around the circle.

Perhaps, in most of cases:

a) One class of tempered fifth is twice as far as the other from being pure. In the case of beat rates, one class of tempering fifth beats twice as fast as the other.
b) The values of the tempering are negative, with the exception of the remaining fifth.
c) For the case where the *syntonic* comma is the chosen unit, a remaining *schisma* has to be placed in one of the fifths of the circle or has to be distributed among several fifths of the circle.

Apart from all these principles and decisions there are some considerations given by most authors who have provided solutions to the problem of temperament. These considerations are based on certain symbols or ornaments existing in the scroll which can remain and can be related to the position and the order of the fifths. These symbols refer to the fifth on D#/Eb and C. Some authors have found a reference to the fifth on G.

The only temperament which has no pure fifths is Lindley II. All the rest have three tempered fifths of one size and five tempered fifths of another size, and three other fifths separating them which are pure.

On the other hand, the only author who considers two possibilities for the value of the 'remaining' fifths is Francis in his temperaments II & III, regarding *Cammerton* and *Cornet-ton* temperaments.

According to the decisions taken, temperaments can be grouped as follows:

a) The direction in which the diagram is to be read: showing always the scroll upside, the two possibilities are reading the scroll from left to right and reading from right to left.
b) The starting note.
c) The order of the intervals: all temperaments follow the order of the circle of fifths with the exception of O'Donnell & Lehman III, which follow the order of the chromatic scale.
d) The size of unit: there are three units considered:
    - Pythagorean comma: Lehman II, Ponsford I, Ponsford II, Maunder I, Maunder II, Maunder III, Lucktenberg, Jencka, Lehman III, Lindley-Ortgies I, Lindley-Ortgies II, O'Donnell, Interbartolo-Venturino II, Billeter IIIa, Billeter IIIb,[253] Di Veroli I, Di Veroli II and Glueck.
    - *Syntonic* comma: Lehman I, Mobbs-Mackenzie, Interbartolo-Venturino I, Interbartolo-Venturino III and Jobin.
    - Beat rates: Sparschuh, Zapf, Briggs, Francis II and Francis III.

   Only temperaments Dent I, II and III consider an indeterminate unit, which can be taken as 1/13 of a Pythagorean comma.
e) The value of the tempering units for each kind of fifth around the circle: the most usual values are 1/6 & 1/12, which correspond to the values given to the model by Lindley. Values 1/4, 1/8 & 1/24 are also very often. Other values found are: 1/7, 1/14 & 1/18. For those temperaments based on *syntonic* comma value 1/11 is also found, as well as other values such as 7/132 resulting in the distribution of the previous value among several fifths already tempered, as occurs in Interbartolo-Venturino I & Jobin.

Temperaments given by Francis are the selection of two temperaments from a collection of all possibilities taking into account the direction in which the diagram is to be read and the starting note. He considers the order of the intervals following the order of the circle of fifths, the unit is the beat rate and the values of tempering units 0, 1 & 2, corresponding to the number of squiggles. This is the most systematic method for which all possibilities are considered regarding decisions a) & b). The criterion is fixed regarding decisions c) and d) and all possible values of tempering for the remaining fifth are considered, that is 0, 1 and 2. The final result consists of two transposed temperaments depending on *Cammerton* and *Cornet-ton* pitches.

Temperaments that take into account the ornaments existing in the scroll, that is, C, D#/Eb and G, are: Lehman II (C), Ponsford I & II (G), Lindley-Ortgies II (C in other position), O'Donnell (D#/Eb) & Jobin (D#/Eb).

Likewise, Francis II & III conform with these two arguments although Francis doesn't take into account the principles of his systematic study. Both chosen possibilities are in agreement with these arguments.

[253] Temperaments by Billeter (IIIa & IIIb) are not defined by fractions of a comma but they are a modification of the Lehman II temperament, which is based on the Pythagorean comma.

Summary

The title of Bach’s work, *Well-Tempered Clavier*, suggests that something in relation to temperament has been taken into account at the time of its composition. No definitive explanation of this question is available at this time but several solutions have been published or posted on the Internet. Many of the recent solutions are based on an interpretation of the scroll on the title page of the autograph manuscript. The first author who realizes that this scroll has some meaning other than a decorative function is Andreas Sparschuh in 1999.

‘Table 1: Definition of temperaments’ in ‘Appendix 9: Tables’ includes the definitions of all temperaments provided as solutions for the question, both historical temperaments and new proposals. A summary with the values for the intervals for all proposed temperaments is also given in ‘Table 2: Intervals of temperaments’.

The nomenclature of the temperaments included in these tables follows the criterion defined in the previous survey.

For those temperaments given as an interpretation of the scroll of the manuscript, the following information is included in ‘Table 1: Definition of temperaments’:

a) Direction in which the diagram is to be read.
b) The starting note.
c) The size of unit.
d) The order of the intervals.

Among the authors considered in the previous survey regarding the temperament in Bach's *Well-Tempered Clavier*, a majority give a new proposed temperament as a solution to the question and others take a historical temperament as the best solution (and others give both solutions).

The authors who propose a new temperament are Herbert Kelletat, Herbert Anton Kellner, Bernhard Billeter, John Barnes, Mark Lindley, Andreas Sparschuh, Martin Jira, Michael Zapf, Keith Briggs, John Charles Francis, Bradley Lehman, Thomas Dent, David Ponsford, Richard Maunder, Kenneth Mobbs and Alexander Mackenzie of Ord, George Lucktenberg, Daniel Jencka, Mark Lindley and Ibo Ortgies, John O’Donnell, Miklós Spányi, Graziano Interbartolo and Paolo Venturino, Emile Jobin, Claudio di Veroli and Thomas Glueck.

The authors who give a historical temperament are Herbert Kelletat [Kirnberger III], John Barnes [Werckmeister III], Claudio di Veroli [Tartini - Vallotti], Sergio Martínez [Kirnberger II] and Miklós Spányi [Kirnberger II]. Likewise, Billeter I is a modification of Kirnberger II temperament and Billeter IIa and Billeter IIb are modifications of Kirnberger III temperament.

This is the chronological list of the authors (with the given temperament between brackets):

- Carl Philipp Emanuel Bach (1753)
- Robert Halford Macdowall Bosanquet (1876)
- James Murray Barbour (1947)
- Herbert Kelletat (1960) [Kirnberger III & Kelletat]
- Herbert Anton Kellner (1977) [Kellner]
- Bernhard Billeter (1977) [Billeter I]
- John Barnes (1979, 1) [Werckmeister III]
- John Barnes (1979, 2) [Barnes]
- Claudio di Veroli (1980) [Tartini - Vallotti]
- Herbert Anton Kellner (1981)

- Claudio di Veroli (1981)
- Ralph Leavis (1981)
- Peter F. Williams (1983)
- Mark Lindley (1985)
- Mark Lindley (1993)
- Mark Lindley (1994) [Lindley I & II]
- Andreas Sparschuh (1999) [Sparschuh]
- Martin Jira (2000) [Jira I & II]
- Michael Zapf (2001) [Zapf]
- Paul Simmonds (2003)
- Keith Briggs (2003) [Briggs]
- Sergio Martínez (2003-2004) [Kirnberger II]
- John Charles Francis (2004) [Francis I]
- Bradley Lehman (2005) [Lehman I & II]
- John Charles Francis (2005, 1) [Francis II & III]
- John Charles Francis (2005, 2)
- John Charles Francis (2005, 3)
- Thomas Dent (2005) [Dent I, II & III]
- David Ponsford (2005) [Ponsford I & II]
- Daniel Jencka (2005)
- Richard Maunder (2005) [Maunder I, II & III]
- Carl Sloan (2005)
- Mark Lindley (2005)
- Kenneth Mobbs and Alexander Mackenzie of Ord (2005) [Mobbs-Mackenzie]
- Stuart M. Isacoff (2005)
- Early Music Editor (2005)
- Bradley Lehman (2005, 2)
- George Lucktenberg (2005)
- Daniel Jencka (2006) [Jencka]
- Bradley Lehman (2006, 1) [Lehman III]
- Bradley Lehman (2006, 2)
- Mark Lindley and Ibo Ortgies (2006) [Lindley-Ortgies I & II]
- Bradley Lehman (2006, 3)
- John O'Donnell (2006) [O'Donnell]
- Bradley Lehman (2006, 4)
- Miklós Spányi (2007) [Kirnberger II & Spányi]
- Graziano Interbartolo and Paolo Venturino (2007) [Interbartolo-Venturino I, II & III]
- Peter F. Williams (2007)
- Bernhard Billeter (2008) [Billeter IIa, IIb, IIIa & IIIb]
- Emile Jobin (2008) [Jobin]
- Claudio di Veroli (2009, 1) [Barnes]
- Claudio di Veroli (2009, 2) [Barnes, Di Veroli I & II]
- John Charles Francis (2011)
- Thomas Glueck [Glueck]

There are some other authors whose propositions can not be explained since they are beyond my scope in this paper.[254]

Evaluation according to dissonance theory

Mathematical models for scales, spectra and scores

Before explaining the behaviours of the evaluation method exposed in this work, suitable mathematical models for scales, spectra and scores will have to be defined in order to apply the mathematical procedures exposed in the following sections.

A scale can be easily represented by a set of frequencies or ratios. In general, the representation by ratios is more comfortable since it is common for the whole extension of the register of the instrument, that is, for any octave.[255]

Likewise a spectrum can also be represented easily by a set of partials which have been defined perfectly by their frequency and amplitude.[256]

Finally it is also necessary to model a musical score in a suitable way. The score is modelled on a set of 'chords' or sets of frequencies and a sensory dissonance mathematical model can be applied to it. This implies seeing the score in a total vertical sense, even though the work has a horizontal or contrapuntal texture, such as Bach's *Well-Tempered Clavier*. Each time a melodic movement is produced in one or several voices, a new 'chord' is generated, which is composed of the notes maintained from the previous 'chord' and the new notes resulting from the movement. These 'chords' are associated with different instants that have been associated with them in the model of the score.

Thus each 'chord' of the score must have one temporal parameter associated with it. Evidently the instants of the different 'chords' are not uniformly produced, that is, each 'chord' has different values of duration depending on the movements of different voices. This temporal parameter is expressed in whichever unit related to the duration of any of the note values. The best option is to choose the minimal note value found in the score or another with a very short value in relation to the general rhythm of the work.

On the other hand, the number of notes will be different for each 'chord' depending on the number of voices in each instant. One unison corresponds to a 'chord' with one note and one rest corresponds to a 'chord' with zero notes. In these cases, the dissonance of the 'chord' is equal to the intrinsic dissonance[257] of the note and the total absence of dissonance respectively.[258]

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[254] *Vid.* Section "Others".

[255] Mathematical models for 'good' temperaments are developed in 'Appendix 2: Definition of 'good' temperaments'. There are more mathematical models developed in other works such as LINDLEY, Mark (with R. Turner-Smith): *Mathematical Models of Musical Scales..., op. cit.*; GIRBAU i BADÓ, Joan: "Les Matemàtiques i les escales musicals"..., *op. cit.*; GARCÍA SUÁREZ, Carlos: "Representación algebraica de las escalas musicales"..., *op. cit.*; MARTÍNEZ RUIZ, Sergio: *A mathematical model for musical tunings and temperaments..., op. cit.* Different ways of representating this kind of temperament are also given in the section 'Appendix 3: Representation of 'good' temperaments'.

[256] A more detailed explanation of this can be found in 'Appendix 8: Dissonance theory'.

[257] *Vid.* 'Appendix 8: Dissonance theory'.

[258] A more detailed explanation of this model is developed in MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software..., op. cit.*

Principles of the evaluation method

According to the definition of total dissonance (TD) given by William A. Sethares[259] and used later by myself, the level of dissonance for each piece of the first volume of *The Well-Tempered Clavier* by Johann Sebastian Bach has been calculated.

The Total dissonance consists in a time-weighted average of all values of the dissonance of each "chord" of the piece. Moreover, each value of the dissonance is calculated using a mathematical model of the dissonance curve. This model, also given by Sethares, is the result of a systematic development of previous theories by Reinier Plomp & Willem J. M. Levelt[260] and the sensory dissonance theory by Hermann Ludwig Ferdinand von Helmholtz[261] and Harry Partch.[262]

The dissonance curve shows the sensation produced by a pair of pure tones, one of which has a sliding frequency. The effect of this interference is to establish three differentiated modes of sensory dissonance: beats, roughness and two separated tones. The maximal point of roughness corresponds to the maximal point of dissonance and, according to Helmholtz's sensory theories, this point of the difference frequency coincides with 1/4 of the critical bandwidth of the ear.

The dissonance curve depends on the spectrum of the instrument and the temperament used. Because of this, a specific temperament and a specific spectrum have to be applied to each piece to be evaluated.[263]

For this research, all temperaments given in all known references which have been put forward in the previous survey have been applied to.each piece of the work in order to calculate the total dissonance. Some historical temperaments which are referred in the survey has been also applied in the evaluation.

Likewise, a suitable spectrum for the harpsichord has been applied to the calculation of the total dissonance. The same mathematical model used by Sethares in his researches has been applied to the pieces evaluated in this work:

$$E = \{f_n = (n+1)f_0,\, a_n = 0.9^n;\, 0 \leq n < N;\, N = 9\}$$

where f_n represents the set of peaks of the spectrum and f_0 is the fundamental frequency. This mathematical model represents a spectrum with nine peaks with a decrease in the value of their amplitudes according to the expression of a_n.

Finally, once a table of dissonances of all pieces is obtained, an average for each temperament has to be worked out to obtain a general dissonance level for each case. It is logical to think that the most suitable temperaments will be those for which the lowest values of the dissonance will be obtained, taking into account the average value for all considered pieces of *The Well-Tempered Clavier*.

Available tools

As the reader might imagine, this calculation method requires a suitable computing tool, since it is not possible to proceed by hand. As an example, you can use

[259] *Vid.* SETHARES, William A.: *Tuning,…*, *op. cit.*

[260] PLOMP, Reinier & LEVELT, Willem J. M. "Tonal consonance and critical bandwidth"..., *op. cit.*

[261] HELMHOLTZ, Hermann Ludwig Ferdinand von: *Die Lehre von den Tonempfindungen..., op. cit.*

[262] PARTCH, Henry: *Genesis of a Music. An account of a creative work, its roots and its fulfillments.* New York: Da Capo Press, 1949. Second edition of 1974. ISBN 0-306-80106-X.

[263] *Vid.* "Appendix 8: Dissonance theory". More detailed explanations of dissonance theory can be found in the references quoted in the 'Bibliography'. A brief explanation can also be found in MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software...*, *op. cit.* and MARTÍNEZ RUIZ, Sergio: "La teoría de la sisnancia...", *op. cit.*

the *SpecMusic* programme (or another individually developed). This programme, developed by myself, is devised with C++ language, using the MFC,[264] some Windows API[265] functions, and the software tool Visual C++ (version 6).[266]

This programme applies the explained mathematical models to scales, temperaments or tunings in general, spectrums and scores. This programme supports standard MIDI files and lets one convert MIDI files to score files and also allows the contrary. Moreover, this programme can represent a graphic for wave files.

SpecMusic can work with six types of documents: scales, spectra, scores, standard MIDI files, wave files and ASCII texts. A description of the possibilities and the behaviours for each type of document are described next.

Scales

These documents represent tunings and temperaments and allow one to work with them and apply several calculations.

Files with the extension ".scl" contain the information for scales and have the following structure:

a) A header "Scal" which identifies the type of file.
b) A Boolean value which identifies the kind of scale which distinguishes if the scale can be represented by rational or irrational numbers. Tunings based on harmonic spectra can be represented by rational numbers and they certainly correspond to their exact definition. This is the case for Pythagorean intonation and just intonation. On the other hand, temperaments can only be represented by irrational values. For the former the Boolean value is 1 and for the latter it is 0.
c) An integer variable, which represents the dimension or the number of notes of the scale. Normally this number has the value 12.
d) Data of the scale, i.e. the values of the intervals. This information is kept in this way:
    a. If the scale is rational, intervals are in their rational representation. That is to say, two integers respectively represent the numerator and the denominator of the ratio which defines each interval.
    b. If the scale is irrational intervals are kept only by a decimal number which represents each interval.

For both cases intervals are referred to by the first note of the scale, normally C.

Scales are represented with a column of rational or decimal numbers. Moreover, several representations can be obtained for scales for which several algorithms can be developed and can be run through a dialogue box accessible on the menu of the programme. Different representations are possible:

a) Intervals.
b) Distances (logarithms).[267]
c) Frequencies.

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[264] Microsoft Fundation Classes.

[265] Application Programming Interfaces.

[266] More information about this software tool and its working principles can be found in MARTÍNEZ RUIZ, Sergio: "Desarrollo de un software...", *op. cit.*

[267] Several definitions for distance are explained in 'Appendix 1: Previous mathematical definitions for tunings and temperaments'.

Intervals for rational scales can be represented in two ways: with fractions and with their decimal equivalent numbers. For irrational scales only the second of these is possible.[268]

Intervals, just like distances, can be represented in two ways:

a) By referring to the first note of the scale (normally C).
b) Using consecutive intervals – i.e. semitones for normal twelve note scales.

For representation by distances a unit of measurement must be defined. *SpecMusic* has devised all these units for distances: octaves, equal tempered tones and semitones, thirds, fourths and sixths of a tone, *merides* (Sauveur), Holder commas, *savarts* (Savart), *heptamerides* (Sauveur), centitones (Yasser), milioctaves (Herschel), Savart logarithms, cents (Ellis), *decamerides* (Sauveur), Pythagorean commas, *syntonic* commas and *schismata*.[269]

Finally, it has been taken into account that a frequency reference (pitch or frequency for A4) must be fixed for representation by frequencies.

Different functions of conversion accessible through the programme menu give all these various representations for a particular scale. There is no graphical representation for scales in *SpecMusic* at the moment.

On the other hand, a tool accessible from the menu of the programme permits different operations with note intervals as a function of their tuning.

To obtain scales, one of the following methods can be used:

1) Introducing the values of the intervals (fractions or decimals, depending on the scale) using the programme editor. The representation of the intervals is referred to by the first note of the scale.
2) Obtaining the related scale for a spectrum.[270]
3) Using several algorithms to create scales implemented in *SpecMusic*. With this algorithm the following types of scales can be obtained at the moment:[271]
    a. Pythagorean intonation,
    b. Just intonation or physical scales,
    c. Meantone temperaments.

Spectra

These documents represent spectra according to the model described in the appendix and allows us to work with them and apply several calculations using them.

Files with the extension ".spt" contain the information for spectra and have the following structure:

a) A header "Spec" which identifies the type of file.
b) An integer variable which represents the dimension or the number of peaks or partials that define the spectrum.

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[268] A representation with the exact definition of some irrational scales (such as temperaments) could be designed using *n*-th roots and products but this device has not been implemented to date. Other scales for which an exact definition of their intervals is unknown can only be represented by decimal numbers.

[269] These unities are defined in 'Appendix 1: Previous mathematical definitions for tunings and temperaments'.

[270] *Vid.* "Appendix 8: Dissonance theory", section 'Related spectra and scales', and SETHARES, William A.: *Tuning,…*, *op. cit.*

[271] The method of creation of scales used in the programme follows certain mathematical models defined by myself, and will be developed in: MARTÍNEZ RUIZ, Sergio: *A mathematical model for musical tunings and temperaments..., op. cit.* For 'good' temperaments, some mathematical models are developed in 'Appendix 2: Definition of 'good' temperaments' but they are not in my programme at the moment.

c) Data of the spectrum, that is, the values of frequencies with their amplitudes. This information is kept in this way: first the values of the frequencies and after this the values of their respective amplitudes.

The representation of spectra consists of two columns with the values of each frequency and their respective amplitude.

One spectrum can be represented in several ways:

a) According to the model defined in the appendix for which a set of partials or spectral lines is represented in two ways:
   a. Normalized, that is, frequencies and amplitudes in relation to the values of the first peak. Thus, first values for the spectrum are 1 for both frequency and amplitude.
   b. Applied to a specific frequency and a real value for the amplitude, that is to say, a real spectrum.
b) An equivalent sampled sequence, that is to say, a sequence which represents the values of the amplitude for each value of the frequency according to a certain fixed resolution. This would be equivalent to a Fourier representation of a determinate wave. Frequency as well as amplitude can be normalized for this representation.

The following operations can be applied to a spectrum, all of them accessible through the programme menu:

a) Conversion functions between some of these kind of representations.
b) Calculation of the associated dissonance curve. This representation gives the values of the curve.

Moreover a graphical representation for spectra can be obtained with the programme, as well as for its associated dissonance curve.

To obtain spectra, one of the following methods can be used:

1) Introducing the values of the frequencies and amplitudes using the programme editor.
2) Obtaining a spectrum from a wave file. This method is very limited in *SpecMusic*.

Scores

These documents represent scores according to the appropriate model for applying calculations in relation to dissonance theory.

Files with the extension '.prt' contain the information for scores and have the following structure:

a) A header 'Part' which identifies the type of file.
b) An integer variable which represents the number of 'chords' which compose the complete score, according to the mathematical model described.
c) Data of the score, that is to say, the sequence of 'chords' which are kept in this way:
   a. An integer variable which represents the number of notes that compose the 'chord'.
   b. Notes of the 'chord' with this format:
      i. An integer value which represents the note, that is, a numerical representation of a musical scale.
      ii. An integer value which represents the octave or acoustical index.
   c. An integer value which represents the duration of the 'chord'.

The 'chords' of the score can be represented in several ways:

a) Notes which compose the ‘chord’ represented variously as:
   a. Integer numbers (position of the note in the musical scale) with their acoustical index or octave using the English or French-Belgium representation.
   b. Names of the notes in several languages: English, German, French, Spanish and Italian are possible in *SpecMusic*.
   c. Integer numbers according to the representation in standard MIDI files are also possible with *SpecMusic* in decimal and hexadecimal formats.
b) Ratios of the intervals of the ‘chord’ represented in two ways:
   a. Ratios referred to the lowest note of the ‘chord’.
   b. Ratios between consecutive notes of the ‘chord’.

In both cases, all possible representations of intervals described for scales are possible with *SpecMusic*.

The following operations can be applied to scores:

a) Different functions of conversion between each of these kinds of representation which are accessible through the programme menu. Of course, for the frequency representation, a scale is expected by the user, including the value of the frequency of reference or pitch.
b) Calculation of the associated dissonance score. This representation gives the values obtained after the calculation of the dissonance score. The value of the Total dissonance (TD) is added at the end of the sequence of the numbers obtained. For this calculation a scale and a spectrum is expected by the user.

Moreover, a graphical representation for scores is given by the programme with a table, including for the dissonance score, that is to say, a graphical representation of the values obtained.

In the frequency representation or even in the calculation of the dissonance curve an application of an adaptive algorithm instead of a scale can be applied to the score. For this case also a spectrum is expected by the user. In *SpecMusic* only Sethares' algorithm is implemented at the moment. It also includes some arguments or parameters which can be introduced by the user. This kind of algorithm has no sense for the study of temperaments in *Well-Tempered Clavier* by Johann Sebastian Bach.

To obtain scores, one of the following methods can be used:

1) Introducing the values of the score using the programme editor.
2) Conversion from a MIDI file. This is the most useful way since MIDI files can be obtained more easily.

MIDI sequences

Standard MIDI files can be read in *SpecMusic*. These files have the extension “mid”. The sequence is shown with a sequence of events, which are described shortly with a text.[272] Notes in message events can be represented according to any of the possible ways used for scores.

Apart from the information of headers, events are represented by a temporal variable (Delta Time) in ticks followed by the explanation of the event which corresponds to that instant. The unit tick corresponds to the temporal unit used in

[272] It is not possible to give a more extended and clearer explanation for the structure of standard MIDI files. More information about these files can be obtained from the references given in the ‘Bibliography’.

standard MIDI files and its real temporal value is determined by the information given in the file header.[273]

Moreover, the sequence can also be represented as a string of bytes making up the file, which consists of hexadecimal numbers composed of two digits.

Conversion functions for all representations are found in *SpecMusic*. A function to change the tuning is also given in the conversion menu for MIDI sequences. This function allows one to change the tuning of the notes in the MIDI sequence. Changes are shown in the editor but, at this moment, they cannot be saved in a new file.

A multimedia player has been designed into the programme in order to listen to the MIDI sequences. These tuning changes can be listened too but they cannot be saved in a new file. A recording device is also planned in order to obtain MIDI sequences from an electronic keyboard.

A graphic representation of a standard MIDI file would be very useful as would a way to edit them in a graphic format. It is not possible to include this capacity in the programme: this would require very complex programming. In fact, this is unnecessary since many programmes to edit and show standard MIDI files can be found in the market.

The best way to obtain MIDI files is to use suitable software. The Internet is a very good source to find them. Using *SpecMusic*, it is also possible to obtain a MIDI file from a score edited using the same programme.

Wave files

Wave files can be read in *SpecMusic*. These files have the extension “wav”. The sequence is shown by a sequence of integer numbers which correspond to samples of the wave.[274]

The only function of conversion contained in *SpecMusic* for wave files is the normalization of the amplitude and the application of one value for the amplitude.

Moreover a multimedia player has been included in the programme in order to listen to the sound of the wave file. A recorder is also planned in order to obtain wave files with a microphone

A graphical representation of wave files is also included in *SpecMusic*. When a wave file is edited with the programme, a graphical representation of this can be shown in a second window.

To obtain wave files, another software programme can be used. The Internet is a very good source to find these files, as is the case with MIDI files. Moreover, using *SpecMusic*, it is also possible to obtain a wave file from a spectrum edited with the same programme.

Texts

Apart from all of these documents, a very useful and common document has been incorporated in *SpecMusic*: a simple ASCII text or Word processor. This document is useful in order to save any of the other documents in a text format readable with other software. In particular, this capability is very useful for keeping results of

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[273] It is not possible to give a complete explanation of the exact structure of standard MIDI files and their events and other parameters. More information about this can be obtained from the references given in the ‘Bibliography’.

[274] It is not possible to give a complete explanation of the exact structure of wave files and their parameters. More information about this can be obtained from the references given in the ‘Bibliography’.

several conversion procedures contained in the programme since all documents are always kept in a default format.

Conversion of documents

Several functions have been included in *SpecMusic* in order to obtain certain documents from other documents:

a) Conversion from a spectrum to a scale. This is the so-called related scale. The inverse procedure is also possible theoretically but is not included in *SpecMusic* at the moment.[275]
b) Conversion from a wave file to a spectrum. For this procedure, an algorithm for the detection of peaks has been included but its functionality is very limited. Previously the Fourier transform had to be applied to the wave (using the FFT[276] algorithm).
c) Conversion from a spectrum to a wave file using the IFFT[277] algorithm to obtain its inverse Fourier transform, the inverse procedure of the former.
d) Conversion from a standard MIDI file to a score. This capability is very useful since the best way to edit a score is to use a music editor and convert it into a MIDI file (or perhaps to find the MIDI file on the Internet).
e) Conversion from a score to a MIDI file, the inverse procedure of the former.

Results obtained

The evaluation method explained in the last paragraphs has been applied to all temperaments given in all known references in the previous survey apart from other historical temperaments also referred to there. For this purpose the *SpecMusic* programme has been used.

For each of the temperaments considered and for each piece of the twelve preludes and fugues in major keys of the first volume of Bach's *Well-Tempered Clavier*, one value for the Total dissonance has been obtained.[278] Thus the average and the standard deviation of data obtained for each temperament and for all considered pieces have been calculated. This final result can be useful for evaluating all given temperaments in relation to their application to Bach's work.

The values obtained indicate the level of dissonance produced by each temperament in each piece and in the whole work. At first, if the main objective is decreasing the dissonance, temperaments which give the lowest results for the average of the Total dissonance through the whole work will be considered the most suitable solutions to the question of temperament in Bach's *Well-Tempered Clavier*.

The 'Table 3: Results obtained' given in 'Appendix 9: Tables' shows the results obtained for all considered temperaments as well as the values of the average and the standard deviation for each one. Their evaluation allows us to obtain some conclusions in relation to the central question of this paper.

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275 Both procedures are given in SETHARES, William A.: *Tuning,…*, *op. cit.* and SETHARES, William A.: "Local consonance...", *op. cit.*

276 Fast Fourier Transform.

277 Inverse Fast Fourier Transform.

278 MIDI files of all preludes and fugues of the first volume of Bach's *Well-Tempered Clavier* (apart from twelve of the twenty-four pieces of the second volume) can be downloaded from: *A Johann Sebastian Bach Midi Page* [online]. In: <http://www.bachcentral.com/>. [January 2011].

Evaluation of results obtained

According to the results of the 'Table 3: Results obtained' in 'Appendix 9: Tables', the minimum value for the average of the Total dissonance is obtained for Kirnberger II temperament.[279] Nevertheless, the standard deviation is the highest for all temperaments evaluated.

Other good results for the Total dissonance have been obtained for the following cases: Spányi, Briggs, Billeter I, Kelletat & Mobbs-Mackenzie. Quite good results are also obtained for the following cases: Kirnberger III, Tartini - Vallotti and Kellner. The worst results are obtained for Neidhardt 1732 - 5th circle XI & Lehman III. Other temperaments yield positive values for the Total dissonance: Neidhardt 1732 - 3rd circle IV, Jira I, Francis I, Lehman II, Ponsford II, Interbartolo-Venturino II, Interbartolo-Venturino III, Billeter II & Di Veroli II. In these cases, the dissonance is higher than that obtained with Equal Temperament.[280] The rest of the values obtained (the majority) fall between -2 and 0.

On the other hand, the highest values for the standard deviation are obtained for the following temperaments: Kirnberger II (the highest), Briggs, Francis I, Interbartolo-Venturino I, Interbartolo-Venturino II, Interbartolo-Venturino III & Jobin.

The second best result for the Total dissonance is obtained for the Spányi temperament, which consists of a modification of the Kirnberger II temperament. It is logical, therefore, that the result is very similar. This supports the idea that Kirnberger II can be chosen as a suitable temperament for the *Well-Tempered Clavier* since Kirnberger II is a much too simple temperament if variations considered by Spányi are not taken into account. A very good result is also obtained for the Kirnberger III temperament, the other temperament created by this pupil of Bach's, and for Kelletat, an author who suggested that one of the temperaments published by Kirnberger was really Bach's own. Curiously, the standard deviation is quite high for all three cases.

Among all temperaments for which a very good value for the Total dissonance is obtained (Kirnberger II, Spányi, Briggs, Billeter I, Kelletat and Mobbs-Mackenzie), only those by Briggs and Mobbs-Mackenzie are based on an interpretation of the diagram included in the cover of the autograph manuscript. Nevertheless, the majority of these interpretations yield negative values for the Total dissonance.

For historical temperaments considered in this work, the results obtained are also quite good, with the exception of two of the three Neidhardt temperaments.

Temperaments and key signatures in Bach's works

Apart from the evaluation method exposed in the previous sections, another evaluation has been made according to a calculation given by John Charles Francis.[281]

For this, a table with the frequency of the occurrence of sharps / flats in Bach's clavier and organ works has been created starting from the catalogue of Bach's works.[282] This table contains the number of movements or pieces which are written in

[279] The same result was obtained in my previous works: MARTÍNEZ RUIZ, Sergio: *Desarrollo de un sotware..., op. cit.* and MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia...", *op. cit.*
[280] According to the expression of the Total Dissonance (*Vid.* "Appendix 8: Dissonance theory"), the results are normalized in order to obtain a null value when Equal Temperament is applied to the work.
[281] FRANCIS, John Charles: "The Esoteric Keyboard Temperaments...", *op. cit.* & FRANCIS, John Charles: "The Esoteric Keyboard Temperaments...", *op. cit.*
[282] DÜRR, Alfred; KOBAYASHI, Yoshitake; BEIßWENGER, Kirsten (eds.): *Bach-Werke-Verzeichnis: Kleine Ausgabe 1998 (BWV 2a).* Leipzig: Breitkopf & Härtel, 1998, 490 pages. ISBN: 3765102490.

each of the 12 possible key signatures (major or minor tonalities) for harpsichord, organ and both instruments (the sum of the other two numbers).[283]

With this information, the correlation between temperaments and the frequency of the occurrence of sharps / flats in Bach's clavier and organ works has been calculated in order to obtain a relation between specific temperaments and the number of used accidentals.

The correlation has been worked out according to the expression expounded in 'Appendix 1: Previous mathematical definitions for tunings and temperaments'. The covariance between the frequency of the occurrence of key signatures and the temperament has been calculated in the following way: the frequency of key signatures has been related to the size of the major third expressed in cents above the tonic of the the major key. For example, for clavier works, the number of works without accidentals in their key signature (105) has been related to the size in cents of the interval C-E (C is the major key without accidentals in its key signature); the number of clavier works with 1 sharp in their key signature (97) has been related to the size in cents of the interval G-B (G is the major key with 1 sharp in its key signature); etc... The correlation has been calculated for clavichord works, for organ works and for both together. The results can be found in 'Table 5: Correlations' in 'Appendix 9: Tables'. Taking only into account the correlations for clavichord works (and also taking into account that those for organ works are similar), the best results are obtained for the following temperaments: Kirnberger III, Tartini - Vallotti, Neidhardt 1732 - 3rd circle II - Small city / Neidhardt 1724 I - Village, Sorge 1744 III, Kelletat, Lindley I, Lindley II - Average Neidhardt, Billeter IIb & Billeter IIIb. The best result (-0.98) corresponds to Lindley II - Average Neidhardt temperament and the second best result (-0.97) is obtained for the Kirnberger III temperament. This result coincides with the best result obtained by John Charles Francis[284] since temperaments by Lindley are not evaluated in his work. The other results are fall between -0.95 and -0.96. On the other hand, the worst results are obtained for the following temperaments: Neidhardt 1732 - 5th circle XI, Zapf, Briggs, Francis I, Francis III - Cornet-ton, Ponsford II, Mobbs-Mackenzie, Billeter IIIa and Billeter IIIb. These results fall between -0.39 and -0.69, with the exception of the Ponsford II temperament for which a positive correlation is obtained (0.55). The results for Kirnberger II, Spányi, Jira I and Maunder III fall between -0.70 and -0.79. The rest of the results obtained (the majority) come between -0.80 and -0.94.

Among temperaments for which a very good value for the Total Dissonance is obtained, i.e. Kirnberger II, Spányi, Briggs, Billeter I, Kelletat, Mobbs-Mackenzie, Kirnberger III, Tartini - Vallotti and Kellner, a very good value for the correlation betwen them and the frequency of the occurrence of sharps / flats in Bach's clavier works is also obtained for the Kirnberger III, Tartini - Vallotti & Kelletat temperaments.

Conclusions

After the obtention of the values of the Total Dissonance for all temperaments evaluated in this work for all studied pieces from the *Well-Tempered Clavier* by Johann Sebastian Bach, and after the obtention of the values of the correlation between all temperaments evaluated and the frequency of the occurrence of sharps / flats in Bach's Clavier works, some conclusions can be obtained for the present research.

[283] *Vid.* 'Table 4: Key signatures in Bach's clavier and organ works' in 'Appendix 9: Tables'.
[284] *Vid.* FRANCIS, John Charles: "The Esoteric Keyboard Temperaments...", *op. cit.*, figure 37, page 45; FRANCIS, John Charles: "Review of the article 'Bach's extraordinary temperament...'", *op. cit.*, page 2.

The results obtained show that the Total Dissonance for all preludes and fugues in major keys from the *Well-Tempered Clavier* by Johann Sebastian Bach can be minimized using Kirnberger II temperament. This is the same conclusion which was obtained in my previous work,[285] although a large number of new temperaments has been taken into account in the current evaluation. Other historical temperaments could be analyzed and perhaps a better result could be obtained but this question is beyond our scope in this paper.

One might think that Kirnberger II is too simple to be considered as a practical temperament but it is also interesting to notice that the modifications introduced by Miklós Spányi[286] in 2007 yield a new and very similar temperament for which the results from the point of view of sensory dissonance are also very good. The Total Dissonance obtained when applying Spányi's temperament represents the second best value for all temperaments evaluated. Moreover, this result is supported by Herbert Kelletat,[287] who suggested that one of the temperaments published by Kirnberger was really Bach's own. Likewise, Billeter I temperament is very similar to Kirnberger II. These latter two temperaments also yield a very good result from the point of view of sensory dissonance.

Using the Kirnberger III temperament, another temperament created by this pupil of Bach's, a very good result is also achieved. One of the strongest correlations between this temperament and the frequency of the occurrence of sharps / flats in Bach's Clavier works is obtained as well.[288] Billeter IIa and Billeter IIb temperaments are also modifications of Kirnberger III temperament and a very good correlation is obtained for Billeter IIb as well.

The only inconvenience using one of these three temperaments (Kirnberger II, Spányi & Kirnberger III) is that a very high value of the standard deviation results, especially for both Kirnberger II and Spányi. At first, this means that some of the pieces can sound very dissonant although others are tuned very correctly from the point of view of sensory dissonance. Nevertheless, the results of 'Table 3: Results obtained' show that the values of the dissonance for each piece of the work are not too elevated (on the contrary, they are quite good) and the minimum value of the average is due above all to the very reduced values of two of the twelve evaluated pieces, specifically BWV 846 (C major) and BWV 860 (G major). On the other hand, the minimum value of the average for the Kirnberger III temperament is basically due to BWV 846 (C major), BWV 856 (F major) and BWV 866 (B flat major), although these results are not as meaningful as those for Kirnberger II.

In the Kirnberger II temperament, the fifths D-A and A-E are reduced by 1/2 of a *syntonic* comma whereas the rest of the fifths of the circle - with the exception of F#-C# which includes the *schisma* - remain pure. This distribution results in the obtention of the pure major thirds C-E, G-B and D-F#, as well as the pure minor thirds E-G and B-D. Curiously, both thirds which compound the triads on C and G are pure, that is, the tonic of the two pieces for which the best values are obtained for the Total dissonance. Moreover, both major thirds of the triads on G and D are also pure, that is, the dominant of the same two pieces. It is probably not chance that these skills are related to the results obtained for the pieces in C and G major and for Kirnberger II.

[285] *Vid.* MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia...", *op. cit.*
[286] *Vid.* SPÁNYI, Miklós: "Kirnberger's Temperament...", *op. cit.*
[287] *Vid.* KELLETAT, Herbert: *Zur Musikalischen Temperatur..., op. cit*.
[288] The strongest correlation is obtained for Lindley II – Average Neidhardt and Lindley III temperaments but the Total Dissonance obtained when applying any of these temperaments is not very good.

In the Kirnberger III temperament, the fifths C-G, G-D, D-A and A-E are reduced by 1/4 of a *syntonic* comma whereas the rest of the fifths of the circle - with the exception of F#-C# which includes the *schisma* - remain pure. This distribution results in the obtention of the pure major third C-E as well as other almost pure major thirds, that is, F-A and G-B which are enlarged by 1/4 of a *syntonic* comma. Moreover, there are also two other major fifths enlarged by 1/2 of a *syntonic* comma, that is, Bb-D and D-F#. On the other hand, there are no pure minor thirds but there exist two minor thirds enlarged only by 1/4 of a *syntonic* comma.

In addition, the following statements can be taken into account with reference to the Kirnberger III temperament:

- In C major tonality, the fifth on the tonic is reduced by 1/4 of a *syntonic* comma, the third on the tonic is pure and the minor third of the triad on the tonic is almost pure (enlarged by only 1/4 of a *syntonic* comma).
- In F major tonality, the fifth on the tonic is pure, the third on the tonic is almost pure and the minor third of the triad on the tonic is also almost pure (both enlarged by only 1/4 of a *syntonic* comma).
- In B flat major tonality, the fifth on the tonic is pure and both the third on the tonic and the minor third of the triad on the tonic are enlarged by 1/2 of a *syntonic* comma.

These three tonalities have the best triads on the tonic and it is probably not by chance either that these tonalities coincide with the tonalities of the pieces which the best values of the Total Dissonance are obtained for.

Finally, it is necessary to notice that the best results have been obtained for historical temperaments which are similar to Kirnberger II and no relation between Bach's diagram and these temperaments can be found at the moment. The only exceptions are Briggs and Mobbs-Mackenzie temperaments although they do not really represent very meaningful results. Thus, on the one hand, it is easy to think that temperaments defined as an interpretation of the gliph drawn in the autograph manuscript are not suitable enough for *The Well-Tempered Clavier*. On the other hand, if the developed theories regarding the squiggle are considered sufficiently credible, one might think that Bach's *Well-Tempered Clavier* is "more dissonant" than expected. Nevertheless, it is also possible that the method of evaluation is not sophisticated enough as a consequence of its limitations. So several improvements of the procedure can be taken into account in order to devise a better version of the software.

Further investigation

As was brought out in my previous work,[289] some improvements in the evaluation system used in this research can be taken into account since they could be deflected from a real solution. Some of these limitations are shown below:

- First, it has to be taken into account that only a selection of all historical temperaments have been evaluated in this work and, moreover, only the preludes and fugues in major tonalities of the first volume of J. S. Bach's *Well-Tempered Clavier* are studied. A complete evaluation of the problem would include the study of the preludes and fugues in minor tonalities and the 24 pieces from the second volume of the work. The rest of the known historical temperaments would also have to be analysed. The analysis could even be extended to other clavier works by Johann Sebastian Bach.

[289] *Vid.* MARTÍNEZ RUIZ, Sergio: "La teoría de la disonancia...", *op. cit.*

- Numerous preoccupations are involved in the creative process of the composer, not only the control of the number of intervals with certain qualities or dissonance level. Thus one cannot expect the results regarding the level of dissonance to be free of variations due to those other factors.[290]

- Any of the temperaments proposed may differ significantly from Bach's real temperament for which there is no evidence of his exact structure.[291]

- Some musical resources used with expressive finality can increase the value of the dissonance. For example, the intentional use of dissonant intervals, either those crossing the wolf fifth[292] or other augmented and diminished intervals (especially fourths and fifths), or other rhetorical figures according to the so-called doctrine of affections,[293] which achieves its greatest splendour in the Baroque period. Other resources like fast or low passages can also increase the dissonance. Likewise, other dissonances due to the contrapuntal style, that is, nonchord tones,[294] can also increase the level of dissonance. These notes are ornaments and they would have not been given the same prominence.[295] The method of evaluation would have to be modified taking into account these variations in the dissonance. The modifications would influence the definition of the mathematical model of the dissonance, which would have to be set up in the software tool and also in the provision of certain extra analytical tools in the programme. On the one hand, a previous musicological and automatic analysis of the work should detect any musical resource used as an expressive finality such as rhetorical figures, false intervals or other passages which could increase the value of the dissonance. This would be a very complex but a very interesting work. On the other hand, more analytical tools should find nonchord tones.

- The dissonance of some intervals is reduced because its notes are not played simultaneously. This phenomenon is caused by the attenuation of the sound wave and it is especially evident in the harpsichord. This effect should be solved by redefining the mathematical expression of the dissonance of the interval defining certain parameters of weight which could be different for each note of the chords depending on the instant in which each note is played.

- The use of a mathematical model of the spectrum constitutes an evident deviation from the reality. Apart from this, a unique model has been considered for the complete register of the instrument whereas the timbre changes with the frequency, that is to say, it is different for each note of the keyboard. Likewise, the mathematical expression of the dissonance of the interval should be redefined to include a new spectral model which should also be redefined as a function of the fundamental frequency.

In order to improve the results, the evaluatuion method should include these modifications of the mathematical expression of the dissonance, either of the interval or the total dissonance. They should be set up in the software and the programme should also provide several tools to detect the various musical resources which influence the calculation.

On the other hand, some effects which have in fact been taken into account in the current research are: the duration of an interval which results in a major or minor

[290] *Vid.* BARNES, John: "Bach's keyboard temperament"..., *op. cit.*

[291] *Vid.* BARNES, John: "Bach's keyboard temperament"..., *op. cit.*

[292] These intervals cannot appear in Bach's *Well-Tempered Clavier* where the use of a good temperament is implicit.

[293] *Affektenlehre* in German.

[294] Anticipations, neighbor tones, escape tones, passing tones, suspension or suspended notes, retardations and *appoggiaturas*.

[295] A null or a minor weight should be applied for nonchord tones in the calculation of the dissonance.

contribution to the dissonance; and the difference of octaves between the notes of an interval or set of intervals.

Finally, another method proposed by William A. Sethares[296] could be set up in the programme in order to find a new temperament which would adapt to the work as much as possible in relation to the mathematical model proposed for the dissonance and the total dissonance. This could be achieved by applying a gradient descent method.

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[296] *Vid.* SETHARES, William A.: *Tuning..., op. cit.*, chapter 9, p. 202.

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Appendix 1: Previous mathematical definitions for tunings and temperaments

Interval ratio

The interval ratio r is defined as the division of the frequencies of its constitutive notes f_2 and f_1:

$$r = f_2 / f_1$$

Then, if $r>1$, the interval is ascendant; on the contrary, if $r<1$, the interval is descendant. The unison corresponds to the particular case for which $r=1$.

Interval distance

In order to define a measurement closer to the musical concept of the distance of an interval, a function likewise depending on the frequencies of its notes has to be defined, taking into account the following properties which it would have to carried out:

1) It has to be defined in the ambit of all possible intervals, that is, in the set of the positive real numbers (without zero).

2) It should be increasing in the ambit in which it is defined.

3) It should be continuous in the ambit in which it is defined.

4) It should be positive for ascendent intervals and negative for descendent ones.

5) Taking into account properties 1) and 3) and Bolzano's theorem, the unison should have a null distance associated with it.

6) Multiplication of intervals should correspond with the sum of associated distances; and division of intervals should correspond with the difference of distances.

One function which carries out all these properties is the logarithm:

$$\delta = \log_a r = \frac{\log r}{\log a}$$

where r is the ratio of the interval, a is the ratio of another interval, which is taken as a measure of distance, and δ is the distance of the interval.

The absolute distance is the absolute value of the logarithm, in which the directon of the interval is not considered.

$$\delta = \left| \frac{\log r}{\log a} \right|$$

The value of a can be defined as a specific fraction of an octave, that is:

$$a = 2^{1/N}$$

and the definition of distance is modified in the following way:

$$\delta = N \frac{\log r}{\log 2} \text{ or } \delta = N \left| \frac{\log r}{\log 2} \right|$$

where N is the number of these units which complete an octave.

Depending on the value of N, several units of measurement can be defined for the distance. Below several common examples are enumerated, some of them atributted to an author:

N=1, octaves

N= 6, tempered tones

N=12, tempered semitones

N=18, thirds of a tone

N=24, quarters of a tone
N=36, sixths of a tone
N=43, *merides* (Sauveur)
N=53, commas (Holder)
N=300, savarts (Savart)
N= 301, *heptamerides* (Sauveur)[298]
N=600, centitones (Yasser)
N=1000, milioctaves (Herschel)
N=1200, cents (Ellis)
N=3010, *decamerides* (Sauveur)[299]

Of course, the value of *a* can take any value, which need not necessarily be a division of the octave. For example, any of the common commas (Pythagorean comma, *syntonic* comma or *schisma*).

The cent is the most common unity of measure of intervals and it was used by Alexander John Ellis[300] for the first time. This unit is equivalent to a hundredth of a semitone in Equal Temperament, that is:

$$a = \left(\sqrt[12]{2}\right)^{1/100}$$

Taking this last value for *a*, the following expression is obtained for the definition of an interval measured in cents:

$$\delta = 1200 \frac{\log r}{\log 2}$$

Distance between temperaments

In order to compare temperaments, several mathematical expressions for the distance between them have been defined.[301] These definitions are based on the application of the typical statistical measurements. Below some of these definitions are described:

Euclidian distance: This is the "ordinary" distance between two temperaments which are considered as points whose distance could be measured with a ruler. The mathematical expression is given by the Pythagorean theorem, that is to say, the square root of the sum of the square power of all frequency differences between both temperaments:

$$\Delta_{12} = \sqrt{\sum_{i=0}^{N} \left(\delta_i^2 - \delta_i^1\right)^2}$$

where δ_i^1 and δ_i^2 are the distances in cents of the intervals of temperaments 1 and 2.

Correlation: This value measures the level of dependence between two temperaments. Its expression is obtained by dividing the covariance of the two tunings by the product of their standard deviations.

The expected value *E(x)* of a certain random value or observed data values *x* corresponds to the mean or average of those values and can be calculated as follows:

$$\bar{x} = E(x) = \frac{1}{N} \sum_{i=0}^{N} x_i$$

298 The *heptameride* is defined as 1/7 of a *meride*, that is, 301=43*7.
299 The *decameride* is defined as 1/10 of a *heptameride*,that is, 3010 = 43*7*10.
300 HELMHOLTZ, Hermann Ludwig Ferdinand von: *Die Lehre von den Tonempfindungen..., op. cit.* English edition translated by Alexander John Ellis. London, 1875.
301 The same definitions can also be applied to tunings.

On the other hand, the covariance σ_{xy} is a measure of how much two random variables x and y change together and the variance σ_x^2 is a special case of the covariance when the two variables are identical. Likewise, the variance represents the average of the deviations with respect to the expected value which is squared. The square root of the variance is called standard deviation. The mathematical expressions for all these definitions are:

$$\sigma_{xy} = E[(x-E(x))(y-E(y))] = \frac{1}{N}\sum_{i=1}^{N}(x_i - E(x))(y_i - E(y))$$

$$\bar{x}^2 = \sigma_x^2 = E\left[(x-E(x))^2\right] = \frac{1}{N}\sum_{i=1}^{N}(x_i - E(x))^2$$

$$\sigma_x = \sqrt{E\left[(x-E(x))^2\right]} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - E(x))^2}$$

Finally, the development of the expression of the correlation is as follows:

$$r_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y} = \frac{E[(x-E(x))(y-E(y))]}{\sqrt{E\left[(x-E(x))^2\right]}\sqrt{E\left[(y-E(y))^2\right]}} = \frac{\frac{1}{N}\sum_{i=1}^{N}(x_i - E(x))(y_i - E(y))}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - E(x))^2}\sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i - E(y))^2}} =$$

$$= \frac{\sum_{i=1}^{N}(x_i - E(x))(y_i - E(y))}{\sqrt{\sum_{i=1}^{N}(x_i - E(x))^2}\sqrt{\sum_{i=1}^{N}(y_i - E(y))^2}}$$

Thus the mathematical expression for the correlation between two temperaments 1 and 2 can be developed in the following way:

$$r_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} = \frac{\sum_{i=1}^{N}\left(\delta_i^1 - E(\delta^1)\right)\left(\delta_i^2 - E(\delta^2)\right)}{\sqrt{\sum_{i=1}^{N}\left(\delta_i^1 - E(\delta^1)\right)^2}\sqrt{\sum_{i=1}^{N}\left(\delta_i^2 - E(\delta^2)\right)^2}}$$

Distance correlation: According to the expression of the correlation between two temperaments, the correlation distance can de defined with the following expression:

$$\Delta_{12} = 1 - r_{12} = 1 - \frac{\sigma_{12}}{\sigma_1\sigma_2} = 1 - \frac{\sum_{i=1}^{N}\left(\delta_i^1 - E(\delta^1)\right)\left(\delta_i^2 - E(\delta^2)\right)}{\sqrt{\sum_{i=1}^{N}\left(\delta_i^1 - E(\delta^1)\right)^2}\sqrt{\sum_{i=1}^{N}\left(\delta_i^2 - E(\delta^2)\right)^2}}$$

Finally, other definitions for the distance are possible and their mathematical expressions are:

Chebyshev distance:

$$\Delta_{12} = \max_i \left|\delta_i^2 - \delta_i^1\right|$$

City block distance:

$$\Delta_{12} = \sum_{i=o}^{N}\left|\delta_i^2 - \delta_i^1\right|$$

Canberra distance:

$$\Delta_{12} = \sum_{i=o}^{N} \frac{\left|\delta_i^2 - \delta_i^1\right|}{\delta_i^1 + \delta_i^2}$$

Minkowski distance:

$$\Delta_{12} = \left(\sum_{i=o}^{N} w_i \left|\delta_i^2 - \delta_i^1\right|^{\lambda} \right)^{1/\lambda}, \ w_i = 1, \lambda = 3$$

Appendix 2: Definition of ‘good’ temperaments

Definition and classification

‘Good’ temperaments are a subset of the Baroque irregular circular temperaments. In cyclical temperaments there is not a wolf fifth and it is possible to modulate to all tonalities. Likewise, irregular temperaments are those where more than one fifth of the circle is different from the rest. On the contrary, equal temperament is that where all fifths are equal and the octave is divided into 12 parts or equal semitones.[302]

There are two fundamental methods to obtain ‘good’ temperaments:

a) Temperaments based on Pythagorean intonation where the Pythagorean comma is eliminated and distributed among the circle of fifths;
b) Temperaments based on meantone temperaments where one *syntonic* comma is eliminated and distributed among the circle of fifths.[303] Moreover a remaining *schisma* is tempering one of the fifths or distributed among several fifths. Then the eliminated *diesis* would correspond to a *schisma*.

In all cases, the remaining intervals which have been eliminated are distributed among several fifths of the circle in a non uniformly way.

‘Good’ temperaments based on Pythagorean intonation

For the definition of a “good” temperament, we can start from the definition of the Pythagorean intonation, for which any interval is a combination of *n* fifths and *m* octaves:

$$\Delta(n,m) = n \log\frac{3}{2} + m \log 2$$

If we desire to delimit the range of the intervals to an octave, we have to make $\Delta(n) = 0$ and *m(n)* becomes:

$$m(n) = -E\left\lfloor \frac{\log\frac{3}{2}}{\log 2} n \right\rfloor$$

[302] More information can be obtained from a number of references, for example: GOLDÁRAZ GAÍNZA, José Javier: *Afinación y temperamento en la música occidental.* Madrid: Alianza Editorial, 1992, 143 pp. ISBN 84-206-8558-5. Re-edition: *Afinación y temperamentos históricos*. Madrid: Alianza Editorial, 2004, 272 pp. ISBN 84-206-6546-0. More references are quoted in the ‘Bibliography’. This mathematical explanation for ‘good’ temperaments will be also given in MARTÍNEZ RUIZ, Sergio: *A mathematical model for musical tunings and temperaments.., op. cit.*

[303] There are some cases where more than one *syntonic* comma is considered in the definition of the temperament.

We can then express the number of the octaves as a function of the number of fifths, as follows:

$$\Delta(n) = n\log\frac{3}{2} + m(n)\log 2$$

And we can calculate any interval as a function of a single parameter.

As the reader knows, there is no value of *n* for which we can obtain a null interval, that is, an octave. One good approximation is obtained for the value of twelve fifths, for which it is easy to demonstrate that we have to reduce the interval to seven octaves. This is the same as taking $n_0 = 12$ and $m_0 = -7$ in the following expression:

$$0 = n_0 \log\frac{3}{2} + m_0 \log 2 - \xi_0$$

where ξ_0 represents the value of the *comma pitagorico*, which is easy to calculate from the previous expression:

$$\xi_0 = \log\frac{531441}{524288} \quad (23.46 \approx 24 \text{ cents})$$

At this moment, the expression for Equal Temperament can be obtained using the value of the *comma pitagorico* and modifying the expression for Pythagorean tuning. The basic idea consists of dividing the value of the *comma pitagorico* among twelve fifths of the circle. We can express this idea mathematically in this way:

$$\Delta(n) = n\left(\log\frac{3}{2} - \frac{\xi_0}{n_0}\right) + m\log 2$$

The previous equation represents the idea of equality of the temperament, making all fifths the same size. That is, the excess of the *comma pitagorico* is compensated by reducing all fifths by the same ratio. Then it is logical to think that fifths will be reduced by a twelfth part of the *comma pitagorico*. The value of fifths can be expressed as follows:

$$q(n) \equiv q_{ET} = \log\frac{3}{2} - \frac{\xi_0}{n_0}$$

On the other hand, we assume that the idea of equality of the temperament consists of a division of the octave into several identical parts. This idea can be expressed mathematically as follows:

$$\Delta(n) = \frac{n}{n_0}\log 2$$

It is possible, and is not very difficult, to demonstrate the equivalence between both representations of Equal Temperament but this is beyond our objectives in this work.

Nevertheless, the main objective in this paragraph was to obtain a general definition of “good” temperaments. An expression of them can be calculated easily from the expression of Equal Temperament since their definition starts with this concept.

The value of the *comma pitagorico* is divided uniformly among all fifths of the circle for Equal Temperament. Nevertheless, the same thing can be done but in a non-uniform way. Using this process, a different size for each fifth of the circle would be obtained. The way to do this division has not been determined and there exist lots of possibilities. This is precisely the reason why there are lots of “good” temperaments.

Then the value of each fifth of the circle can be expressed in this way:

$$q(n) = \log\frac{3}{2} - \alpha(n)\xi_0$$

where each fifth has been reduced to a fraction of the *comma pitagorico*, which is different for each position of the circle. Of course lots of values for $\alpha(n)$ are null.

$\alpha(n)$ can be defined as a sequence of n_0 rational values, that is:

$$\alpha(n) \in \mathbf{Q}, 1 \le n \le n_0$$

According to the last expression, the tempering amount for every fifth can be expressed by:

$$\Delta q(n) = -\alpha(n)\xi_0$$

where

$$q(n) = Q + \Delta q(n)$$

and

$$Q = \log\frac{3}{2}$$

is the pure fifth.

In order to calculate the general expression for "good" temperaments, the value of each fifth of the circle must include its tempering amount as a function of the parameter that represents the position of the interval along the circle:

$$\Delta(n) = \sum_{i=1}^{n} q(i) + m(n)\log 2 = \sum_{i=1}^{n}\left[\log\frac{3}{2} - \alpha(i)\xi_0\right] + m(n)\log 2$$

$$\Delta(n) = n\log\frac{3}{2} - \xi_0\sum_{i=1}^{n}\alpha(i) + m(n)\log 2$$

where m has already expressed as a function of the number of fifths. Taking into account the value of the last interval of the circle:

$$\Delta(n_0) = 0$$

$$n_0\log\frac{3}{2} + m_0\log 2 - \xi_0\sum_{i=1}^{n_0}\alpha(i) = 0$$

$$n_0\log\frac{3}{2} + m_0\log 2 = \xi_0\sum_{i=1}^{n_0}\alpha(i)$$

and taking also into account the same value in Pythagorean intonation:

$$n_0\log\frac{3}{2} + m_0\log 2 = \xi_0$$

the following condition must be fulfilled by the set $\alpha(n)$ of fractions of the Pythagorean comma:

$$\sum_{i=1}^{n_0}\alpha(i) = 1$$

which is the condition for working out the set $\alpha(n)$ of fractions of the *comma pitagorico*.

Typically these numbers are equal to a fraction that can be expressed as a multiple of one 12th of the *comma pitagorico*, *i. e.:*

$$\alpha(n) = \frac{n}{12}$$

In the most typical cases,

$$n \in \{0, 1, 2, 3\}$$

and

$$\alpha(n) \in \left\{0, \frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right\}$$

Finally, in order to evaluate the thirds for every temperament, it is useful to work out the tempering of thirds around the whole circle in the following way:

Starting with the expression of a generic third:

$$t(n) = \sum_{i=n}^{n+3} q(i) - m(n)\log 2 =$$

$$= \sum_{i=n}^{n+3} \log\frac{3}{2} - \alpha(i)\xi_0 - m(n)\log 2 = \log\frac{81}{64} - \xi_0 \sum_{i=n}^{n+3} \alpha(i)$$

and defining

$$t(n) = T + \Delta t(n)$$

where

$$T = \log\frac{5}{4}$$

the tempering for every third is:

$$\Delta t(n) = t(n) - T = \log\frac{81}{64} - \xi_0 \sum_{i=n}^{n+3} \alpha(i) - \log\frac{5}{4} = \log\frac{81}{80} - \xi_0 \sum_{i=n}^{n+3} \alpha(i)$$

$$\Delta t(n) = \varphi_0 - \xi_0 \sum_{i=n}^{n+3} \alpha(i)$$

'Good' temperaments based on meantone temperaments

Another typical model for 'good' temperaments is that defined through the value of the *syntonic* comma. Some pure thirds can be obtained using this temperament. Starting again from the definition of Pythagorean intonation:

$$\Delta(n) = n\log\frac{3}{2} + m(n)\log 2$$

where

$$m(n) = -E\left\lfloor \frac{\log\frac{3}{2}}{\log 2} n \right\rfloor,$$

applying the definition of meantone temperaments and applying the interval approximation theory, a pure interval with the form:

$$\Delta(n_0, m_0, p_0) = n_0 \log\frac{3}{2} + p_0 \log\frac{5}{4} + m_0 \log 2$$

can be approximated by the expression of Pythagorean intonation, where m_0 can also be obtained as a function of n_0 and p_0. The usual case is to assign a pure major third, with ratio 5/4, which can be obtained by assigning the following values:

$$n_0 = 0,\ m_0 = 0,\ p_0 = 1$$

resulting in the following expression:

$$\log\frac{5}{4} = n_1 \log\frac{3}{2} + m_1 \log 2 - \varphi_0$$

where φ_0 is the *syntonic* comma and which is obtained for $n_1=4$ and $m_1=-2$. Then:

$$\varphi = \log\frac{81}{80} \text{ (21.51} \approx \text{22 cents)}$$

Using the value of *syntonic* comma, meantone temperament can be defined by distributing the value of the comma uniformly among all fifths of the circle, that is:

$$\Delta(n) = n\left(\log\frac{3}{2} - \frac{\varphi_0}{n_1}\right) + m(n)\log 2$$

This temperament is known as meantone temperament of 1/4 of a comma (the comma here is the *syntonic* comma) since the fraction of the comma which tempers fifths is $1/n_1$, that is, 1/4.[304] Likewise it is known only as meantone temperament since it is the main and the most usual case.

For the last value of the circle of fifths we have:

$$\Delta(n_0) = n_0\left(\log\frac{3}{2} - \frac{\varphi_0}{n_1}\right) + m_0\log 2$$

$$0 = n_0\left(\log\frac{3}{2} - \frac{\varphi_0}{n_1}\right) + m_0\log 2 + \rho_0$$

where ρ_0 is the *diesis* and the interval:

$$\log\frac{3}{2} - \frac{\varphi_0}{n_1} + \rho_0$$

is the wolf fifth.

On the other hand, for the last value of the circle of fifths in Pythagorean intonation:

$$\Delta(n_0) = n_0\log\frac{3}{2} + m_0\log 2$$

$$0 = n_0\log\frac{3}{2} + m_0\log 2 - \xi_0$$

Thus the following equivalence can be assimilated:

$$\frac{n_0}{n_1}\varphi_0 - \rho_0 = \xi_0$$

and the remaining value for the *diesis* in the irregular temperament can be obtained easily:

$$\rho_0 = \frac{n_0}{n_1}\varphi_0 - \xi_0$$

Replacing the values of n_0 and n_1, the value of the diesis is the difference between 3 *syntonic* commas and one Pythagorean comma:

[304] This is a specific case of a more generic set of meantone temperaments which can be defined according to the following mathematical expression:

$$\Delta(n) = m(n)\log 2 + n(\log\frac{3}{2} + \tilde{\varphi}_1),\ \tilde{\varphi}_1 = \frac{\bar{n}_3}{\bar{n}_2 + n_1\bar{n}_3}\varphi_0,\ \xi_0 = \log\xi_0',\ \varphi_0' = \frac{81}{80},\ n_1 = 4.$$

The expression $\frac{\bar{n}_3}{\bar{n}_2 + n_1\bar{n}_3}$, that relates both remainders as a function of certain parameters, identifies the kind of temperament and is called the comma ratio or fraction of a comma. Each temperament receives its name from the ratio which defines it. A more generic explanation of these temperaments can be obtained in several references given in the 'Bibliography' and the development of their mathematical definition will be given in: MARTÍNEZ RUIZ, Sergio: *A mathematical model for musical tunings and temperaments..., op. cit.*

$$\rho_0 = 3\varphi_0 - \xi_0$$

$$\rho_0 = \log\left(\frac{\varphi_0'^3}{\xi_0'}\right) = \log\left[\left(\frac{81}{80}\right)^3 : \frac{531441}{524288}\right] = \log\frac{128}{125} \text{ (41.06} \approx \text{41 cents)}$$

Temperaments based on the *syntonic* comma can be defined in the same way as 'good' temperaments based on the *comma pitagorico* have been defined. Starting from the meantone temperaments equation instead of the Equal temperament, a 'good' temperament can be defined distributing the value of the *syntonic* comma in a non uniformly way among all fifths of the circle. The way to do this division has not been determined and there exist lots of possibilities. This is precisely the reason why there are lots of "good" temperaments.

Then the value of each fifth of the circle can be expressed in this way:

$$q(n) = \log\frac{3}{2} - \alpha(n)\varphi_0$$

where each fifth has been reduced with a fraction of the *syntonic* comma which is different for each position of the circle. Of course lots of values for $a(n)$ are null.

$\alpha(n)$ can be defined as a sequence of n_0 rational values, that is:

$$\alpha(n) \in \mathbf{Q}, 1 \leq n \leq n_0$$

According to the last expression, the tempering amount for every fifth can be expressed by:

$$\Delta q(n) = -\alpha(n)\varphi_0$$

where

$$q(n) = Q + \Delta q(n)$$

and

$$Q = \log\frac{3}{2}$$

is the pure fifth.

In order to calculate the general expression for "good" temperaments, the value of each fifth of the circle must include its tempering amount as a function of the parameter that represents the position of the interval along the circle:

$$\Delta(n) = \sum_{i=1}^{n} q(i) + m(n)\log 2 = \sum_{i=1}^{n}\left[\log\frac{3}{2} - \alpha(i)\varphi_0\right] + m(n)\log 2$$

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) + m(n)\log 2$$

where m has already expressed as a function of the number of fifths.

In the hypothetical case where the temperament would get a just third through n_1 fifths, the following procedure can be applied:

$$\log\frac{5}{4} = n_1\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n_1}\alpha(i) + m_1\log 2$$

and a similar condition would have to be fulfilled by the set $\alpha(n)$ of fractions of the *syntonic* comma:

$$\sum_{i=1}^{n_1}\alpha(i) = 1$$

For the last value of the circle of fifths in a meantone temperament:

$$\Delta(n_0) = n_0\left(\log\frac{3}{2} - \frac{\varphi_0}{n_1}\right) + m_0\log 2$$

$$0 = n_0\left(\log\frac{3}{2} - \frac{\varphi_0}{n_1}\right) + m_0 \log 2 + \rho_0$$

$$0 = n_0 \log\frac{3}{2} - \frac{n_0}{n_1}\varphi_0 + m_0 \log 2 + \rho_0$$

and for the last value of the circle of fifths in Pythagorean intonation:

$$\Delta(n_0) = n_0 \log\frac{3}{2} + m_0 \log 2$$

$$0 = n_0 \log\frac{3}{2} + m_0 \log 2 - \xi_0$$

Thus the following equivalence can be assimilated:

$$\frac{n_0}{n_1}\varphi_0 - \rho_0 = \xi_0$$

On the other hand, for the last value of the circle of fifths in a 'good' temperament based on the meantone temperament:

$$\Delta(n_0) = n_0 \log\frac{3}{2} - \varphi_0\sum_{i=1}^{n_0}\alpha(i) + m_0 \log 2$$

$$0 = n_0 \log\frac{3}{2} - \varphi_0\sum_{i=1}^{n_0}\alpha(i) + m_0 \log 2 - \rho_1$$

where the following equivalences can be obtained from:

$$\frac{n_0}{n_1}\varphi_0 - \rho_0 = \varphi_0\sum_{i=1}^{n_0}\alpha(i) + \rho_1$$

$$\xi_0 = \varphi_0\sum_{i=1}^{n_0}\alpha(i) + \rho_1$$

Proceeding similarly as in temperaments based on Pythagorean intonation, a *syntonic* comma, instead of the Pythagorean comma, is distributed among the completed circle of fifths. Thus, the following condition can be assumed now:

$$\sum_{i=1}^{n_0}\alpha(i) = 1$$

And so the following equivalences can be assumed:

$$\frac{n_0}{n_1}\varphi_0 - \rho_0 = \varphi_0 + \rho_1$$

$$\xi_0 = \varphi_0 + \rho_1$$

Taking into account only the second equivalence, the remaining value for the *diesis* in the 'good' temperament can be obtained easily:

$$\sigma \equiv \rho_1 = \xi_0 - \varphi_0$$

This amount is called a *schisma* and is equal to the difference between the *comma pitagorico* and the *syntonic* comma, that is:

$$\sigma = \xi_0 - \varphi_0 = \log\left(\frac{\xi_0'}{\varphi_0'}\right) = \log\left(\frac{531441}{524288} : \frac{81}{80}\right) = \log\frac{6561}{5120} \text{ (1.95} \approx \text{2 cents)}$$

Apart from the amounts based on fractions of a *comma*, the *schisma* also has to be placed in any position of the circle of fifths in order to close it, but its position does not necessarily have to be the wolf fifth. Several possibilities can be considered:

a) The *schisma* is placed in one of the fifths without tempering by one fraction of a comma. This is the most usual case.

b) The *schisma* is placed in one of the tempered fifths.
c) The *schisma* is distributed among several fifths—in the beginning, between two just fifths.

So the complete definition of a 'good' temperament based on a meantone temperament requires the complete set of the fractions of a comma tempering all fifths of the circle as well as the place of the *schisma*. Consequently, the general expression for 'good' temperaments based on meantone temperament has to be revised including the tempering of the *schisma*. According to the previous possibilities, a general expression which includes all of them can also be defined through another set of rational numbers:

$$\beta(n) \in \mathbf{Q}, 1 \leq n \leq n_0$$

where the following condition is obvious for this case:

$$\sum_{i=1}^{n_0} \beta(i) = 1$$

The resulting expression for a generic fifth would be:

$$q(n) = \log\frac{3}{2} - \alpha(n)\varphi_0 - \beta(n)\sigma$$

and the generic expression for a 'good' temperament based on meantone temperament could be calculated as follows:

$$\Delta(n) = \sum_{i=1}^{n} q(i) + m(n)\log 2 = \sum_{i=1}^{n}\left[\log\frac{3}{2} - \alpha(i)\varphi_0 - \beta(n)\sigma\right] + m(n)\log 2$$

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) - \sigma\sum_{i=1}^{n}\beta(i) + m(n)\log 2$$

Then the total amount of tempering values is:

$$\varphi_0\sum_{i=1}^{n}\alpha(i) + \sigma\sum_{i=1}^{n}\beta(i)$$

that is,

$$\varphi_0 + \sigma = \xi_0$$

For the most typical case, where only one fifth contains the total amount of the *schisma*, the sum could be substituted by a Dirac delta function,[305] that is:

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) - \sigma\delta(n-\tilde{n}) + m(n)\log 2$$

where $\tilde{n}$ represents the position of the *schisma* in the circle of fifths and $0 \leq \tilde{n} \leq n_0$.

As in 'good' temperaments based on Pythagorean intonation, these numbers are typically equal to a fraction that can be expressed as a multiple of one 12th of the *comma pitagorico*, *i. e.*:

$$\alpha(n) = \frac{n}{12}$$

In the most typical cases

$$n \in \{0,1,2,3\}$$

and

[305] This function is defined as:

$$\delta(n-n_0) = \begin{cases} 0, & if\ n = n_0 \\ 1, & if\ n \neq n_0 \end{cases}.$$

$$\alpha(n) \in \left\{0, \frac{1}{12}, \frac{1}{6}, \frac{1}{4}\right\}$$

As a consequence of this, it is very useful and interesting to represent the value of *schisma* in relation to the Pythagorean comma and the *syntonic* comma distributed among all twelve semitones of equal temperament, that is to say, to represent it in relation to a n_0-th (or a twelfth) part of a Pythagorean comma. For this task it is very useful to use logarithms. Then the following equation has to be raised:

$$\sigma = \frac{x}{n_0}\xi_0$$

$$\xi_0 - \varphi_0 = \frac{x}{n_0}\xi_0$$

$$x = n_0(1 - \frac{\varphi_0}{\xi_0})$$

Substituting the values,

$$n_0 = 12, \xi_0 = \log \xi_0', \xi_0' = \frac{531441}{524288}, \varphi_0 = \log \varphi_0', \varphi_0' = \frac{81}{80}$$

the following result can be obtained:

$$x \approx 1$$

So

$$\sigma \approx \frac{\xi_0}{12};\ \xi_0 \approx 12\sigma$$

$$\sigma \approx \frac{\xi_0}{12} = \frac{\sigma + \varphi_0}{12}$$

$$\sigma \approx \frac{\varphi_0}{11};\ \varphi_0 \approx 11\sigma$$

$$\varphi_0 = \frac{11}{12}\xi_0;\ \xi_0 = \frac{12}{11}\varphi_0$$

Finally, according to the previous statements, and since $\sigma \approx \frac{\varphi_0}{11}$, the general expression for 'good' temperaments based on meantone temperament can be rewritten in this way:

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) - \sigma\sum_{i=1}^{n}\beta(i) + m(n)\log 2 \approx$$

$$\approx n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) - \frac{\varphi_0}{11}\sum_{i=1}^{n}\beta(i) + m(n)\log 2 =$$

$$= n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\left[\alpha(i) + \frac{1}{11}\beta(i)\right] + m(n)\log 2$$

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\gamma(i) + m(n)\log 2$$

where

$$\gamma(i) = \alpha(i) + \frac{1}{11}\beta(i)$$

The condition to be carried out by $\gamma(i)$ is:

$$\sum_{i=1}^{n_0}\gamma(i)=\sum_{i=1}^{n_0}\alpha(i)+\sigma\sum_{i=1}^{n_0}\beta(i)\approx\sum_{i=1}^{n_0}\alpha(i)+\frac{1}{11}\sum_{i=1}^{n_0}\beta(i)=1+\frac{1}{11}=\frac{12}{11}$$

and the total amount of tempering values is:

$$\varphi_0\sum_{i=1}^{n_0}\gamma(i)=\frac{12}{11}\varphi_0=\xi_0$$

According to the new generic expression, the tempering amount for every fifth can be expressed as:

$$\Delta q(n)=-\gamma(n)\varphi_0$$

or perhaps, depending on n:

$$\Delta q(n)=-\alpha(n)\varphi_0 \text{ or } \Delta q(n)=-\sigma\approx-\frac{\varphi_0}{11}$$

where

$$q(n)=Q+\Delta q(n)$$

and

$$Q=\log\frac{3}{2}$$

is the pure fifth.

Finally, in order to evaluate the thirds for each temperament, it is useful to work out the tempering of thirds around the whole circle in the following way:

Starting with the expression of a generic third:

$$t(n)=\sum_{i=n}^{n+3}q(i)-m(n)\log 2=$$

$$=\sum_{i=n}^{n+3}\log\frac{3}{2}-\gamma(i)\varphi_0-m(n)\log 2=\log\frac{81}{64}-\varphi_0\sum_{i=n}^{n+3}\gamma(i)$$

and defining

$$t(n)=T+\Delta t(n)$$

where

$$T=\log\frac{5}{4}$$

the tempering for every third is:

$$\Delta t(n)=t(n)-T=\log\frac{81}{64}-\varphi_0\sum_{i=n}^{n+3}\gamma(i)-\log\frac{5}{4}=\log\frac{81}{80}-\varphi_0\sum_{i=n}^{n+3}\gamma(i)$$

$$\Delta t(n)=\varphi_0\left[1-\sum_{i=n}^{n+3}\gamma(i)\right]$$

Appendix 3: Representation of 'good' temperaments

Representation with fractions of a comma

On the one hand, the resulting set of $a(n)$ for a 'good' temperament based on the Pythagorean intonation corresponds to one of the typical representations for the definition of 'good' temperaments—that is to say, the sequence of fractions of a *comma* *pitagorico* which tempers each of the intervals of the circle of fifths.[306] Normally the circle starts in Eb and finishes in D#. The representation can be shown in a table or in a graphical format with a circle.

The following example corresponds to the Neidhardt 1724 I - Village / Neidhardt 1732 – 3rd circle II - Small city temperament:

$$\alpha(n) = \{-1/12, 0, 0, -1/6, -1/6, -1/6, -1/6, -1/12, -1/12, 0, 0, -1/12\}$$

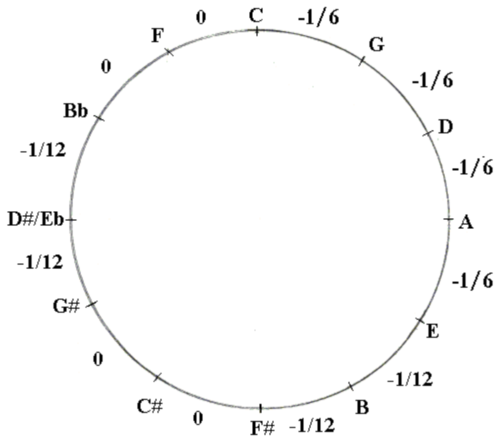

Figure 69 - Neidhardt 1724 I - Village / Neidhardt 1732 – 3rd circle II - Small city temperament

It is easy to prove that the condition for the values of a fraction of a comma is fulfilled in the previous temperament:

$$\sum_{i=1}^{12} \alpha(i) = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 4 * \frac{1}{12} + 4 * \frac{1}{6} = \frac{1}{3} + \frac{2}{3} = 1$$

On the other hand, for the case of 'good' temperaments based on meantone temperament, one can proceed similarly using the corresponding set of fractions of a *syntonic* comma, which tempers each of the intervals of the circle of fifths, that is to say, α *(n)*, as well as the position of the *schisma*, whose value is represented in relation to the *syntonic* comma, that is to say, 1/11. Normally, as in the case of 'good' temperaments based on the Pythagorean intonation, the circle starts in Eb and finishes in D#. The table and the graphical representation are also possible in this kind of temperament.

The following example corresponds to the Kirnberger III temperament:

$$\gamma(n) = \alpha(n) + \frac{1}{11}\beta(n) = \{0, 0, 0, -1/4, -1/4, -1/4, -1/4, 0, 0, -1/11, 0, 0\}$$

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306 It is very common to use the fractions with a negative sign. Then the representation really corresponds to the sequence -$\alpha(n)$.

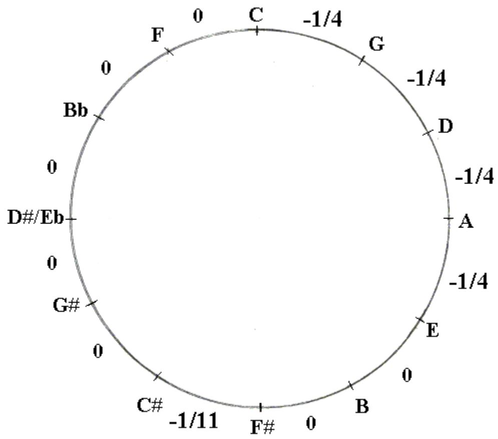

Figure 70: Kirnberger III temperament

It is also easy to prove that the condition for the values of the fractions of a comma is fulfilled in the previous temperament:

$$\sum_{i=1}^{12}\gamma(i)+\frac{1}{11}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{11}=4*\frac{1}{4}+\frac{1}{11}=1+\frac{1}{11}=\frac{12}{11}$$

Likewise, taking into account that:

$$\varphi_0=\frac{11}{12}\xi_0 \text{ and } \xi_0=\frac{12}{11}\varphi_0$$

it is easy to represent a temperament in relation to the other comma. That is:

a) A temperament based on Pythagorean intonation (represented in relation to the *comma pitagorico*) can be represented in relation to the value of the *syntonic* comma:

$$\alpha'(n)=\frac{12}{11}\alpha(n),\ 0<n\le n_0$$

b) And *vice versa*, a temperament based on meantone temperament (represented in relation to the *syntonic* comma) can be represented in relation to the value of the *comma pitagorico*:

$$\gamma'(n)=\frac{11}{12}\gamma(n),\ 0<n\le n_0$$

Representation with ratios and distances

In order to express the definition of temperament through the different ratios of their intervals, the previous logarithm expression has to be converted in a multiplication using the equivalent base-ratios as follows:

$$\Delta(n)=n\log\frac{3}{2}-\xi_0\sum_{i=1}^{n}\alpha(i)+m(n)\log 2$$

$$\delta(n)=\left(\frac{3}{2}\right)^n \xi_0'^{\,-\sum_{i=1}^{n}\alpha(i)}\,2^{m(n)}$$

where:[307]

[307] The best way to work out all intervals of the scale is:

$$\xi_0 = \log \xi'_0$$

$$\Delta(n) = n \log \frac{3}{2} - \varphi_0 \sum_{i=1}^{n} \alpha(i) - \sigma \sum_{i=1}^{n} \beta(i) + m(n) \log 2$$

$$\delta(n) = \left(\frac{3}{2}\right)^n \varphi_0'^{\,-\sum_{i=1}^{n}\alpha(i)} \sigma^{-\sum_{i=1}^{n}\beta(i)} 2^{m(n)}$$

or

$$\delta(n) = \left(\frac{3}{2}\right)^n \varphi_0'^{\,-\sum_{i=1}^{n}\gamma(i)} 2^{m(n)}$$

where:[308]

$$\varphi_0 = \log \varphi'_0$$

Likewise, in order to express the intervals with respect to their equivalent distances, the suitable logarithm expressions given in the previous section[309] have to be applied.

In both cases, intervals have to be ordered from minor to major value so as to order them according to the chromatic scale.

***Representation with* schismata**

As has been shown previously, *schisma* is the difference between *comma pitagorico* and *syntonic* comma.[310] Its value is $\frac{6561}{5120}$ or 1.95 cents (2 cents approximately) and is approximately equal to a twelfth of a Pythagorean comma and to an eleventh of a *syntonic* comma.

This interval corresponds to *diesis* in systems where the whole extension of the circle of fifths has been reduced to only one *syntonic* comma. This is typically the case where a just major third is placed in the diatonic notes of the scale and the rest of the fifths of the circle remain just.

This interval already appears in the work of Bartolomé Ramos de Pareja,[311] for example, as a first approximation of the just intonation in relation to the Pythagorean

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$$\delta(n) = \frac{3}{2}\delta(n-1)\,\xi_0'^{\,-\alpha(n)}\, 2^{-k}$$

where *k=1* if necessary and otherwise *k=0*.

308 The same procedure given for temperaments based on Pythagorean intonation works in this case:

$$\delta(n) = \frac{3}{2}\delta(n-1)\,\varphi_0'^{\,-\gamma(n)}\, 2^{-k}$$

309 *Vid.* 'Appendix 2: Definition of 'good' temperaments'.

310 *Vid.* Section ''Good' temperaments based on meantone temperaments'.

311 RAMOS DE PAREJA, Bartolomei: *Musica practica Bartolomei Rami de Pareia Bononiae, impressa opere et industria ac expensis magistri Baltasaris de Hiriberia MCCCCLXXXII*. Bologna: Baltasar de Hiriberia, May 1482. 2nd edition: June 1482. Reprint by G. Vecchi (ed.): Bologna: Arnalso Forni Editore, 1969. Reprint by Johannes Wolf: *Musica practica Bartolomei Rami de Pareia Bononiae, impressa opere et industria ac expensis magistri Baltasaris de Hiriberia MCCCCLXXXII: Nach den Originaldrucken des Liceo musicale mit Genehmigung der Commune von Bologna.* Publikationen der Internationalen Musikgesellschaft, Beihefte, Heft 2. Leipzig: Breitkopf und Härtel, 1901. Reprint by Rodrigo de Zayas: Madrid: Alpuerto, 1977. English translation by Clement A. Miller: *Bartolomeo Ramis's Musica Practica*, Neuhausen-Stuttgart: Hänssler-Verlag, 1993. Transcription by Jingfa Sun (data entry), Peter Slemon (checking) and Thomas J. Mathiesen (approval): RAMUS DE PAREIA, Bartolomaeus: "Musica practica, prima pars, tractatus primus". In: *Center for the History of Music Theory and Literature. Jacobs School of*

intonation. It also appears in some 'good' temperaments in the 18th century, as can be seen in the previous paragraphs.

Schisma can be taken as a unity of measurement, above all for certain intervals with a very complicated ratio such as the typical remaining intervals: *diesis*, *comma pitagorico*, *syntonic* comma, etc… It has been used specifically for the measurement of the impurity of intervals.

Other authors have used this interval as a unity of measurement for their own tunings or temperaments, for example, John Farey with his schismatic temperaments (1820).[312]

In particular, this unit of measurement was used by some German theorists in Bach's day, most notably Johann Georg Neidhardt and Georg Andreas Sorge to represent different temperaments in relation to just intonation. It is also used today by writers such as Mark Lindley and Ibo Ortgies[313] or John O'Donnell[314] have also used this unit of measurement in order to represent tunings and temperaments.

The following intervals are considered:

- Just third:

$$t = \frac{5}{4}$$

- Just and Pythagorean fifth:

$$q = \frac{3}{2}$$

- Pythagorean third:

$$t_P = q^4 = \frac{(3/2)^4}{2^2} = \frac{81}{64}$$

- Pythagorean comma: difference between twelve fifths and an octave

$$PC = q^{12} = \frac{(3/2)^{12}}{2^7} = \frac{531441}{524288}$$

- *Syntonic* comma: difference between a Pythagorean third and a just third.

$$SC = \frac{t_P}{t} = \frac{q^4}{t} = \frac{(3/2)^4}{2^2} : \frac{5}{4} = \frac{81}{80}$$

- *Schisma*: difference between the Pythagorean comma and the *syntonic* comma.

$$s = PC / SC = \frac{q^{12}}{q^4 / t} = \frac{q^8}{t} = \frac{(3/2)^8}{2^4} : \frac{5}{4} = \frac{6561}{5120}$$

Since $s \approx \frac{PC}{12}$, as has been shown, it can be deduced that *schisma* corresponds to the value of the fraction of the Pythagorean comma which is distributed in each fifth of the circle.[315] Thus the tempered fifth corresponds to the difference between a just fifth and the *schisma*:

Music, Indiana University. Thesaurus Musicarum Latinarum. [online]. In: <http://www.chmtl.indiana.edu/tml/15th/RAMMP1T1_TEXT.html>. [December 2010]. Part I, Treatise 1.

[312] FAREY, John: *On different modes of expressing the magnitudes and relations of musical intervals; with some remarks, in commendation of Professor Fischer's proportionally-tempered douzeave...*, 1820.

[313] LINDLEY, Mark & ORTGIES, Ibo: "Bach-style keyboard tuning"..., *op. cit.*

[314] O'DONNELL, John: "Bach's temperament, Occam's razor...", *op. cit.*

[315] *Vid.* the section ''Good' temperaments based on meantone temperaments'.

$$q_{ET} = q - \frac{PC}{12} = q - s$$

Likewise, the values of commas can be expressed using *schismata* as a unit of measurement, that is:

$$PC \approx 12s$$
$$SC = PC - s \approx 12s - s = 11s$$

The *schisma* constitutes a good unit of measure for the deviation of any interval in relation to just intonation.

For example:

$$t_P = t + SC = t + 11s$$
$$t_{ET} = 4q_{ET} = 4(q - s) = 4q - 4s = t_P - 4s = t + SC - 4s = t + 11s - 4s = t + 7s$$
$$o_{ET,q} = 12q_{ET} = 12(q - s) = 12q - 12s = 12q - PC$$
$$o_{ET,t} = 3t_{ET} = 3(t + 7s) = 3t + 21s$$

Moreover, some properties of this system of measurement must hold:

1. The total amount of tempering for the circle of fifths has to be 12. Of course one does not count both enharmonic intervals represented in the scheme.
2. The total for each chain of four fifths and the resulting major third has to be 11.
3. The total for each column of major thirds has to be 21 units.

This scheme consists of the 16 notes Db to A# represented in four rows with four notes where, moreover, for the first three rows, the last note of each row is repeated at the beginning of the following row. Starting below on the left side, the series continues to the right and ascends following the inclination of the columns. Perhaps, this inclination is not necessary but it is assumed by tradition. Distributing all notes of the series among rows of four notes and adding the first note of the following row, the series of notes in the columns results in series of thirds. The numbers that indicate the amounts of fifths are added between each pair of notes in a row and the amounts of thirds between each pair of adjoining notes in a column. First and last rows represent enharmonic notes. Moreover each column contains the three thirds which provide the octave in the same way that each row contains the four fifths which provide the third.

As examples, the following graphics represent Equal Temperament and one of the Sorge's several temperaments respectively:

```
      C#  1  G#  1  D#  1  A#
     7      7      7      7
    A   1  E   1  B   1  F#  1  C#
   7      7      7      7
  F   1  C   1  G   1  D   1  A
 7      7      7      7
Db  1  Ab  1  Eb  1  Bb  1  F
```

Figure 71 - Equal temperament

```
      C#  0  G#  1  D#  1  A#
     8      9      8      8
    A   1  E   0  B   1  F#  1  C#
   5      4      6      7
  F   0  C   2  G   2  D   2  A
 8      8      7      6
Db  0  Ab  1  Eb  1  Bb  1  F
```

Figure 72 - Sorge 1758 temperament

To sum up, in order to represent the values of the intervals in scales represented or defined by *schismata*, the procedure is:

a) For temperaments based on Pythagorean intonation:

Starting from the general expression of the temperaments:

$$\Delta(n) = n\log\frac{3}{2} - \xi_0\sum_{i=1}^{n}\alpha(i) + m(n)\log 2$$

and taking into account that:

$$\sigma = \frac{\xi_0}{12}$$

if a sequence of real (in general) values *s(n)* represents the deviations in *schismata* for each fifth of the circle, the expression for temperaments can be rewritten as follows:

$$\Delta(n) = n\log\frac{3}{2} - \sigma\sum_{i=1}^{n}s(i) + m(n)\log 2$$

where $s(i) = 12\alpha(i)$.

b) For temperaments based on meantone temperament:

Starting from the general expression of the temperaments:

$$\Delta(n) = n\log\frac{3}{2} - \varphi_0\sum_{i=1}^{n}\alpha(i) - \sigma\sum_{i=1}^{n}\beta(i) + m(n)\log 2$$

and taking into account that:

$$\sigma = \frac{\varphi_0}{11}$$

if a sequence of real (in general) values *s(n)* represents the deviations in *schismata* for each fifth of the circle, the expression for temperaments can be rewritten as follows:

$$\Delta(n) = n\log\frac{3}{2} - \sigma\sum_{i=1}^{n}s(i) + m(n)\log 2$$

where $s(i) = 11\alpha(i) + \beta(i)$.

Likewise, in order to calculate the values of the intervals in scales represented or defined by *schismata*, the previous expression can be used, replacing the value of the *schisma* by its exact value or by its equivalence in relation to either of the commas (which is really an approximation), that is:

$$\Delta(n) = n\log\frac{3}{2} - \frac{\xi_0}{12}\sum_{i=1}^{n}s(i) + m(n)\log 2$$

$$\Delta(n) = n\log\frac{3}{2} - \frac{\varphi_0}{11}\sum_{i=1}^{n}s(i) + m(n)\log 2$$

The most logical procedure is to use the comma by which the temperament is defined.

Appendix 4: Description of some historical 'good' temperaments[316]

Werckmeister

Andreas Werckmeister (30 November 1645, Benneckenstein - 26 October 1706, Halberstadt) [317] presented six tunings and temperaments, the first of which was the first

[316] The values of the intervals of some of the temperaments set out in this section are shown in 'Table 2: Intervals of temperaments' expressed as ratios.

[317] WERCKMEISTER, Andreas: *Musikalische Temperatur..., op. cit.*

of the irregular temperaments in Germany.[318] It became one of the most famous and was imitated by several authors in Germany (Neidhardt, Sorge, Marpurg & Türk).

Werckmeister I: A system of just intonation with twenty notes per octave which will be used as a measurement system or reference system to work out the deviations produced in the rest of his temperaments given in the work. This is an impracticable tuning since this scale requires employing *subsemitonia* (double keys), and this practice is not in use.

Werckmeister II: The usual meantone temperament of 1/4 of a comma, which he denominates *alte*, *falsche* or *unrichtige Temperatur* and rejects.

Werckmeister III (1/4 cp): The Pythagorean comma is distributed among four fifths and the rest of the fifths are just. The objective of Werckmeister was to make the more consonant and usual intervals purer. The temperament gives good major thirds for diatonic notes and Pythagorean thirds for the chromatic notes. This temperament is applied to chromatic tonalities.

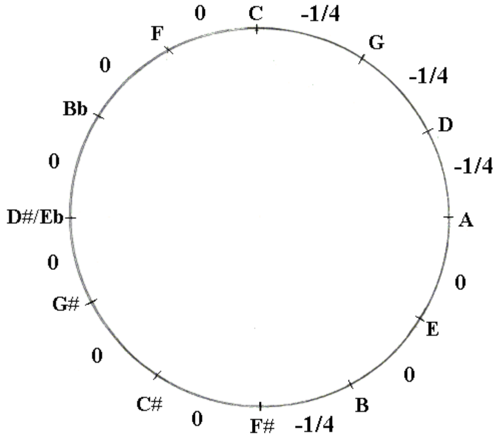

Figure 73 - Werckmeister III temperament

Werckmeister IV (1/3 cp): An alternation between pure fifths and others reduced by 1/3 of a Pythagorean comma. This temperament is applied to diatonic tonalities.

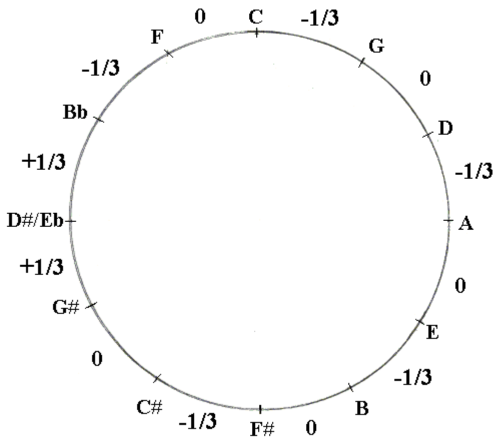

Figure 74 - Werckmeister IV temperament

[318] The first of these temperaments had already appeared in his previous work: WERCKMEISTER, Andreas: *Erweiterte und verbesserte Orgel-Probe..., op. cit.*

Werckmeister V (1/4 cp): An alternation between just fifths and others reduced by 1/4 of a Pythagorean comma, closing the circle with one fifth increased as 1/4 of the same amount.

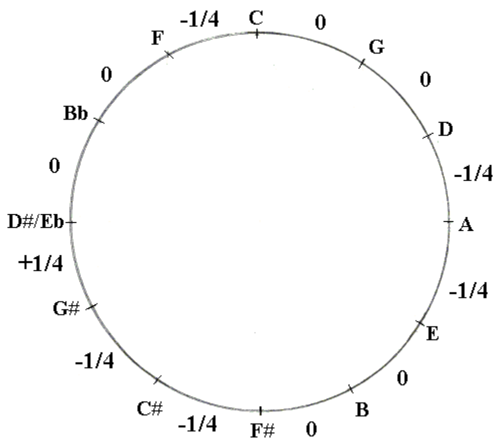

Figure 75 - Werckmeister V temperament

Werckmeister VI (*Temperatur aus dem Septenario*): A closed system with 31 notes per octave. This is not a practical temperament either. Some authors consider it a temperament of -1/7 of a comma.

Kirnberger

Johann Philipp Kirnberger (24 April 1721, Saalfeld - 27 July 1783, Berlin)[319] has three temperaments, the first two of which are already quoted by Friedrich Wilhelm Marpurg (21 November 1718, Seehof (Wendemark, Altmark) - 22 May 1795, Berlin) in 1776.[320] Both date from 1771.

Kirnberger I (1cs): A very simple system where only one fifth is reduced with one *syntonic* comma (Re-La) and another with one *schisma* (F#-C#).

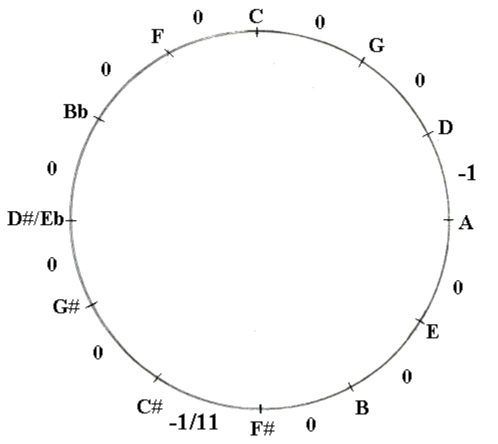

Figure 76 - Kirnberger I temperament

Kirnberger II (1/2cs): An improvement for the previous temperament where the *syntonic* comma is distributed between two fifths (D-A & A-E), apart from the

[319] KIRNBERGER, Johann Phillip: *Die Kunst des reinen Satzes in der Musik..., op. cit.*

[320] MARPURG, Friedrich Wilhelm: *Versuch über die musikalische Temperatur, nebst einem Anhang über den Rameau- und Kirnbergerschen Grundbaß, und vier Tabellen*. Berlin: Barnes, 1776 & Breslau: Johann Friedrich Korn, 1776, 320 pages.

schisma placed in the same place. This is a very archaic temperament, which is strange at the end of the 18th century after the more complicated temperaments by Neidhardt and Sorge. Moreover, it is presented by a disciple of Johann Sebastian Bach. Nevertheless, it was very used at the beginning of the 19th century in Germany, England and Italy. Reduced fifths are acceptable and there are several just thirds in C-E, G-B, D-F#, E-G and B-D. The success of this temperament is due to the facility to put it into practice.

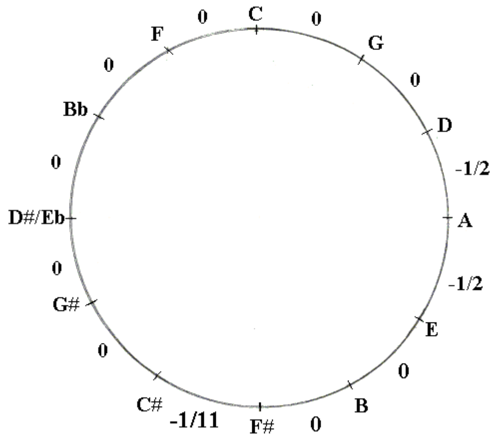

Figure 77 - Kirnberger II temperament

Kirnberger III (1/4cs): Also dated 1771,[321] another improvement on previous temperaments is proposed by Kirnberger in which the *syntonic* comma is distributed among four fifths in order to diminish the imperfection of the reduced fifths between C and E.

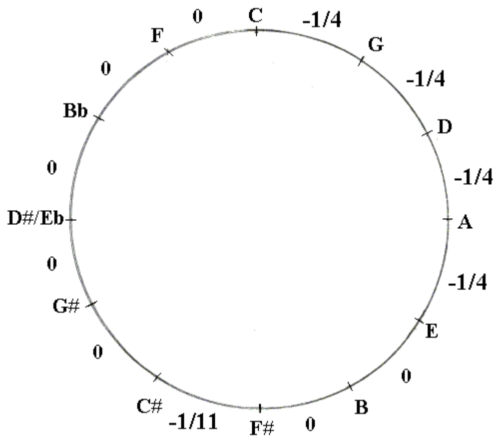

Figure 78 - Kirnberger III temperament

[321] KIRNBERGER, Johann Phillip: Letter to Forkel, [1779]. Published in: BELLERMAN, Heinrich: „Briefe von Kirnberger an Forkel". In: *Allgemeine musikalische Zeitung* ("General music journal"), No. 34 (1871). Transcription in "The letter from Johann Philipp Kirnberger to Johann Nikolaus Forkel, in which the former defines the keyboard temperament that is now known as 'Kirnberger III'". In: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/Kirn_1871.html>. [January 2011]. More information in: "Johann Philipp Kirnberger's definition of the temperament that is now known as 'Kirnberger III'". In: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/K_III.html>. [January 2011].

Vallotti

Father Francesco Antonio Vallotti (11 June 1697, Vercelli - 10 January 1780, Padova),[322] chapelmaster at St. Antonio in Padua, poposes a very simple temperament which reduces the most usual fifths, i.e. the diatonic fifths (F-B), by 1/6 of a *syntonic* comma and the others remain just. The *schisma*, according to the instructions of tuning, is placed between Bb and F.

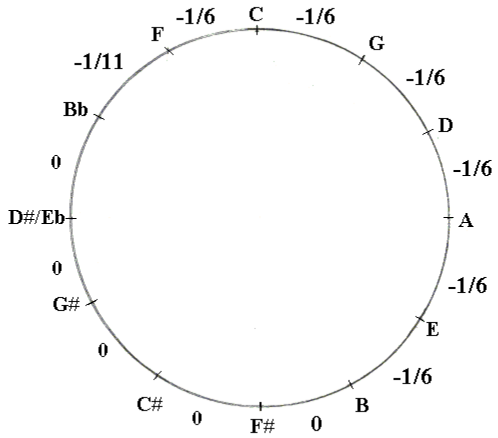

Figure 79 -Vallotti temperament

Nevertheless, this is not the temperament known today as Vallotti. The known version of Vallotti's temperament is due to the (slightly ambiguous) information given by Giuseppe Tartini (12 April 1692, Pirano - 26 February 1770, Padova).[323] This temperament uses the Pythagorean comma instead of the *syntonic* comma and the *schisma* is eliminated. Reduced fifths are placed in the same positions as in Vallotti's original version, that is the diatonic fifths between F and B. Vallotti advises not to use more than four alterations in the key signature. The success of this temperament is due to its simplicity to put into practice.

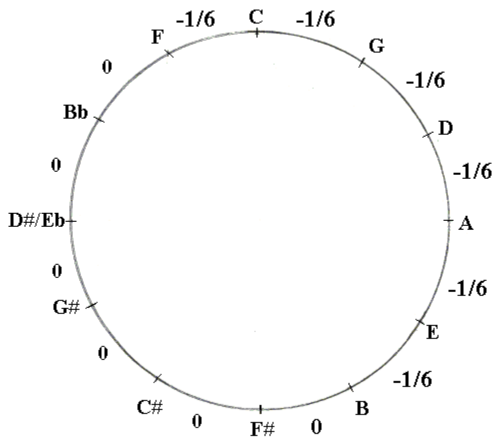

Figure 80 - Tartini - Vallotti temperament

[322] VALLOTTI, Francescoantonio: *Della Scienza Teoretica e Prattica della Moderna Musica..., op. cit*

[323] TARTINI, Giuseppe: *Trattato di Musica secondo la vera Scienza dell' Armonia.* Padova: Giovanni Manfrè, 1754, 175 pages. German translation by Alfred Rubeli: *Traktat über die Musik gemäß der wahren Wissenschaft von der Harmonie*. Düsseldorf: Verlag der Gesellschaft zur Förderung der systematischen Musikwissenschaft, 1966, 397 pages. English translation by Stllingfleet, 1771. Another English translation by F. B. Johnson (ed.): Indiana: PhD diss., University of Indiana, 1985.

Neidhardt

Johann Georg Neidhardt (1680, Bernstadt (Schlesien) - 1 January 1739, Königsberg) proposed 25 temperaments in two treatises published in 1724[324] and 1732.[325] In the treatise of 1724, he proposed four temperaments, which he classified according to their utility (probably depending on the quantity of possible modulations). They are suitable for a village (*Dorf*), a little city (*kleine Stadt*), a big city (*grosse Stadt*) and the court (*Hof*), and the last of these corresponds to the Equal Temperament. On the other hand, in the treatise of 1732, he proposed 21 other temperaments, grouped in the following way:

1) One example of a Phytagorean scheme, where he placed the entire comma into the fifth C-G;

2) 12 temperaments classified as *Quinten-Circul* (fifth-circle), where he intended to distribute variously the Phytagorean comma in order to emphasize the qualities and varieties of fifths;

3) 3 experimental examples, where he intended to show how the major and minor thirds are affected by the distribution of the comma;

4) 5 temperaments classified as *Circul der Tertiarum* (third-circle), where he intended to distribute variously the Phytagorean comma in order to emphasize the qualities and varieties of thirds (more practical exmples).

In this new treatise, he also classified four of the 21 temperaments according to their utility. The first of the fifth-circle temperaments is suitable for the court and also corresponds to the Equal Temperament: the eighth of the fifth-circle temperaments is suitable for a big city; the first of the third-circle temperaments is suitable for a village; and the second of the third-circle temperaments is suitable for a small city. Moreover, the temperament for a village in the 1724 book is the same as the temperament for the small city in the 1732 book; the temperament for a small city in the 1724 book is the same as the temperament in the 1732 book; and the temperament for a big city in the 1724 book does not appear in the 1732 book.

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[324] NEIDHARDT, Johann Georg: Sectio canonis harmonici, zur völligen Richtigkeit der Generum Modulandi. Königsberg, 1724. Facsimile of the section 2.1: Die II. Abtheilung./ Von dem Canone Harmonico./ Das I. Stücke./ Von dem völlig circulirenden Genere,/ und etlichen drauf gegründeten/ Temperaturen. In: Welcome to Harpsichords Wiki! In: <http://harpsichords.pbworks.com/f/Neidhardt_1724_ascii.html>. [June 2010]. Transcription of the section 2.1: Welcome to Harpsichords Wiki! In: <http://harpsichords.pbworks.com/f/Neidhardt_1724_sec2.1.pdf>. [June 2010]; NEIDHARDT, Johann Georg: Gäntzlich erschöpfte, mathematische Abtheilungen des diatonisch-chromatischen, temperirten Canonis Monochordi. Leipzig & Königsberg: C.G. Eckhardt, 1732. Reprint of collected works to be published: Utrecht: The Diapason Press. More information in: LEHMAN, Bradley: "Errata and clarifications for "Bach's extraordinary temperament: Our Rosetta Stone". Early Music, February and May 2005". In: *Johann Sebastian Bach's tuning* [online]. February 2007. In: <http://www-personal.umich.edu/~bpl/larips/errata.html> & <http://www.larips.com>. [January 2011]. Section "Numbering and detail of the 1724 Neidhardt temperaments".

[325] NEIDHARDT, Johann Georg: *Gäntzlich erschöpfte, mathematische Abtheilungen des diatonisch-chromatischen, temperirten Canonis Monochordi.* Leipzig & Königsberg: C.G. Eckhardt, 1732. Reprint of collected works to be published: Utrecht: The Diapason Press. Facsimile of chapters VI, VII & VIII in: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/Neidhardt_1732_sec6-8.pdf>. [June 2010]. Transcription of chapters VI, VII & VIII in: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/Neidhardt_1732_ascii.html>. [June 2010]. More information in: LEHMAN, Bradley: "Johann Georg Neidhardt's 21 temperaments of 1732". In: *Welcome to Harpsichords Wiki!* [online]. In: <http://harpsichords.pbworks.com/f/Neidhardt_1732_Charts.pdf>. [January 2011].

Below is a list with the equivalence between the four temperaments from 1724 and the most usual nomenclature given by James Barbour:[326]

- Neidhardt I: Village.
- Neidhardt II: Small city.
- Neidhardt III: Big city.

A summary of all these equivalences is found in the following table:

1724	1732		Barbour
	Dorf (Village)	Third-circle I	
Dorf (Village)	*Kleine Stadt* (Small city)	Third-circle II	Neidhardt I
Kleine Stadt (Small city)	*Grosse Stadt* (Big city)	Fifth-circle VIII	Neidhardt II
Grosse Stadt (Big city)			Neidhardt III
Hof (Court)	*Hof* (Court)	Fifth-circle I	Equal Temperament

The best-known temperaments are Neidhardt 1732 - 5th circle XI (Neidhardt I) (1/6cp, 1/12cp) and Neidhardt 1732 - 3rd circle II - Small city / Neidhardt 1724 I - Village (Neidhardt III) (1/6cp, 1/12cp). In Neidhardt 1732 - 5th circle XI (Neidhardt I), the purity of diatonic thirds is sought and tonalities are more differentiated than in Neidhardt 1732 - 3rd circle II - Small city / Neidhardt 1724 I - Village (Neidhardt III), which is closer to the Equal Temperament.

1724 Neidhardt's temperaments:

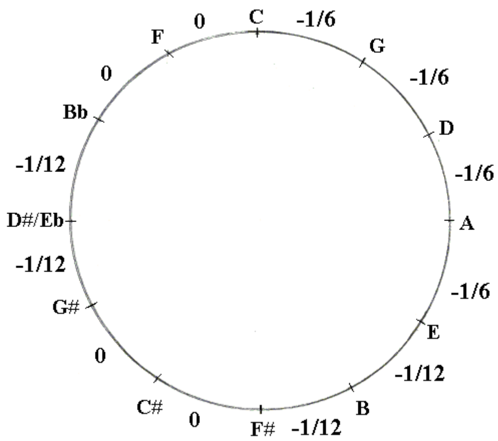

Figure 81 - Neidhardt 1724 I - Village temperament

[326] BARBOUR, James Murray: *Tuning and Temperament..., op. cit.*

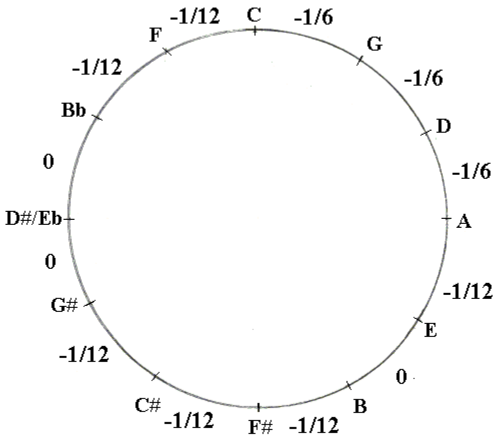

Figure 82 - Neidhardt 1724 II - Small city temperament

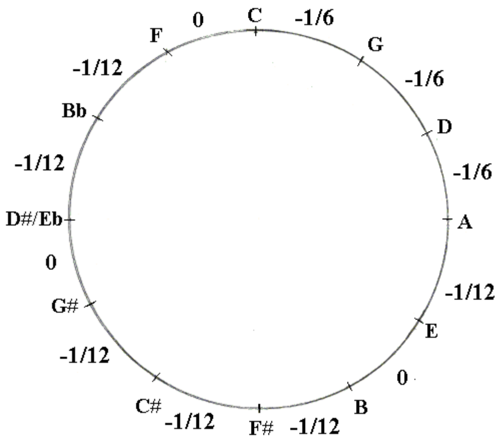

Figure 83 - Neidhardt 1724 III - Big city temperament

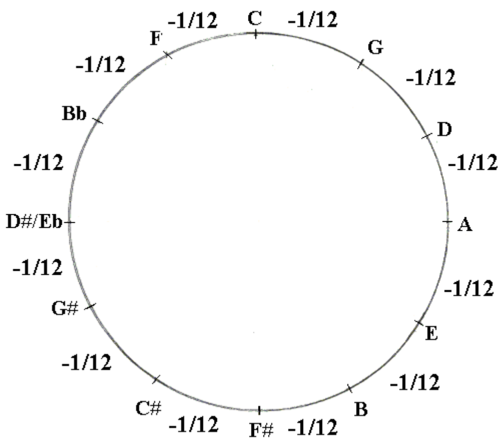

Figure 84 - Neidhardt 1724 IV - Court temperament (Equal Temperament)

1732 - 3rd circle Neidhardt's temperaments:

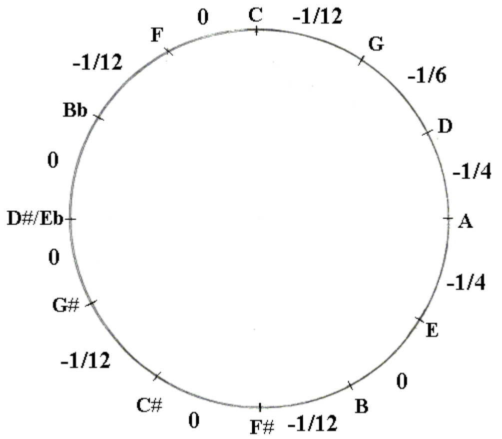

Figure 85 - Neidhardt 1732 - 3rd circle I - Village temperament

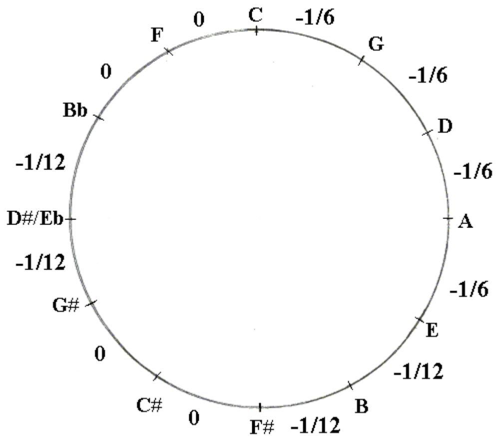

Figure 86 - Neidhardt 1732 - 3rd circle II – Small city temperament

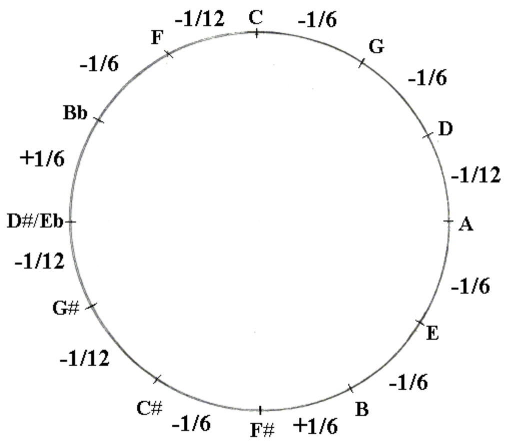

Figure 87 - Neidhardt 1732 - 3rd circle III temperament

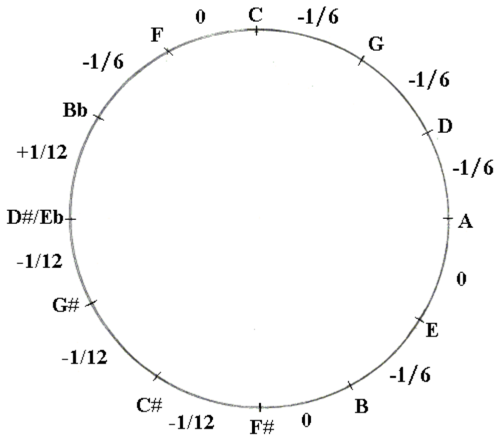

Figure 88 - Neidhardt 1732 - 3rd circle IV temperament

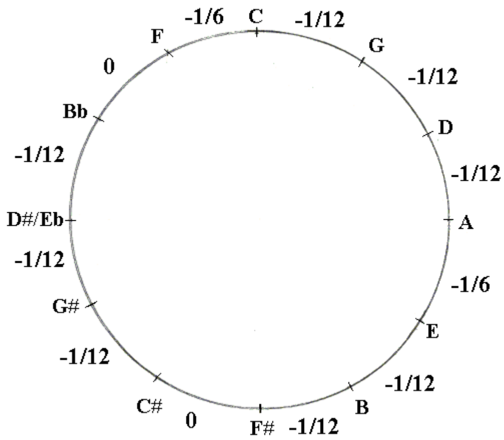

Figure 89 - Neidhardt 1732 - 3rd circle V temperament

1732 – 5th circle Neidhardt's temperaments:

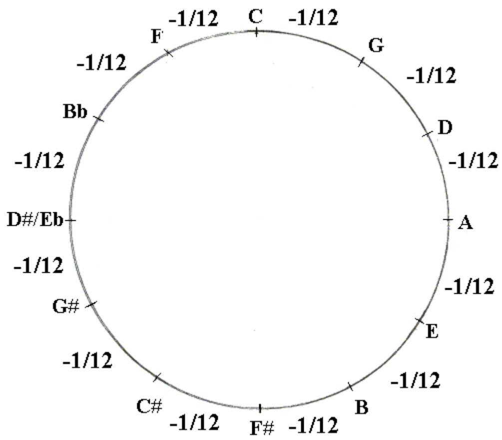

Figure 90 - Neidhardt 1732 - 5th circle I - Court temperament (Equal Temperament)

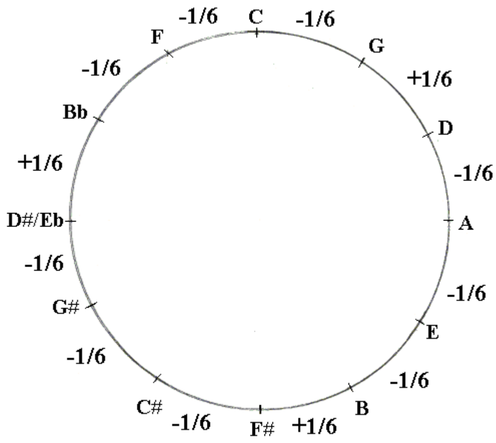

Figure 91 - Neidhardt 1732 - 5th circle II temperament

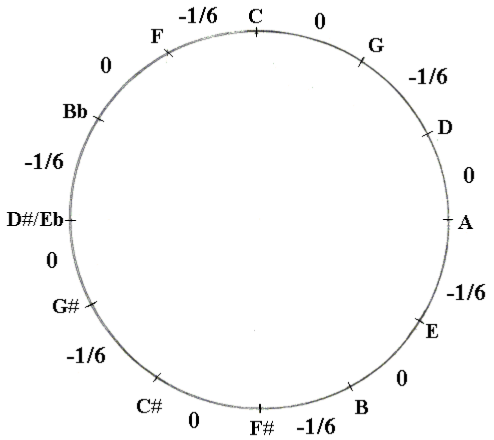

Figure 92 - Neidhardt 1732 - 5th circle III temperament

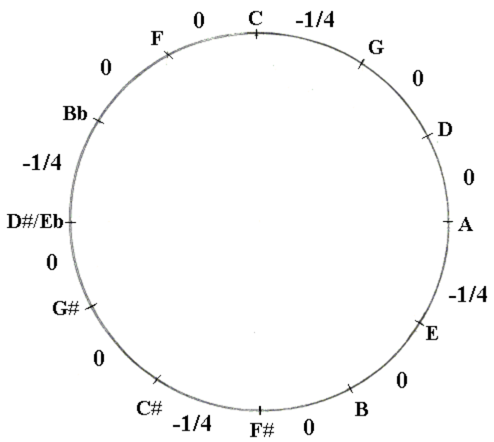

Figure 93 - Neidhardt 1732 - 5th circle IV temperament

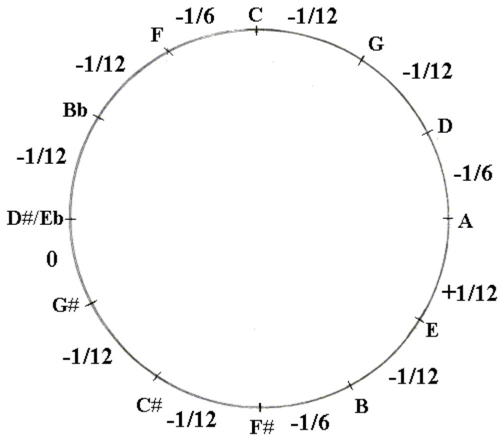

Figure 94 - Neidhardt 1732 - 5th circle V temperament

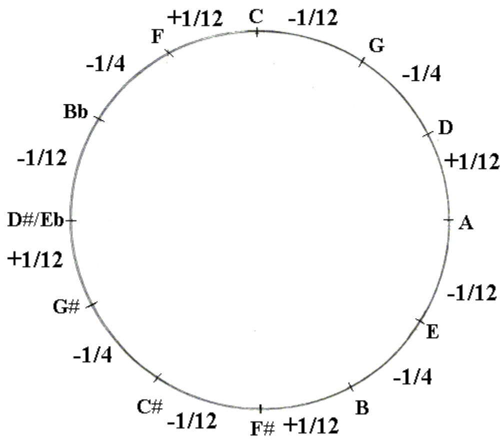

Figure 95 - Neidhardt 1732 - 5th circle VI temperament

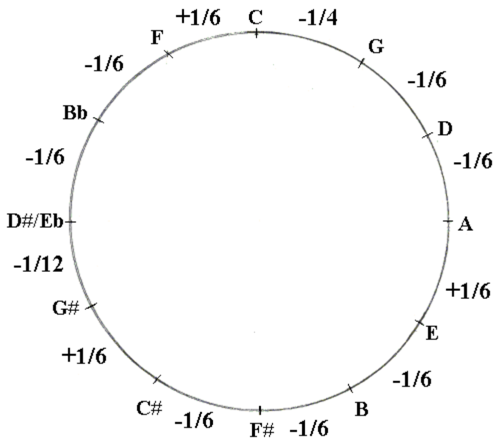

Figure 96 - Neidhardt 1732 - 5th circle VII temperament

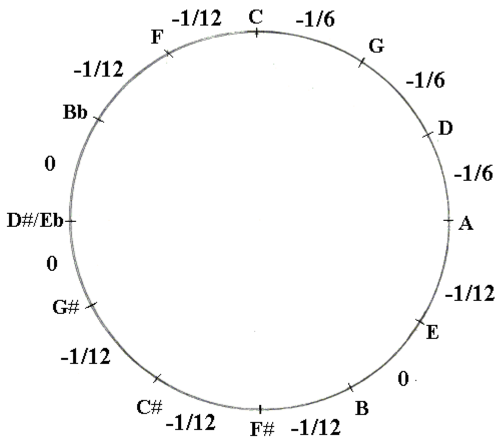

Figure 97 - Neidhardt 1732 - 5th circle VIII - Big city temperament

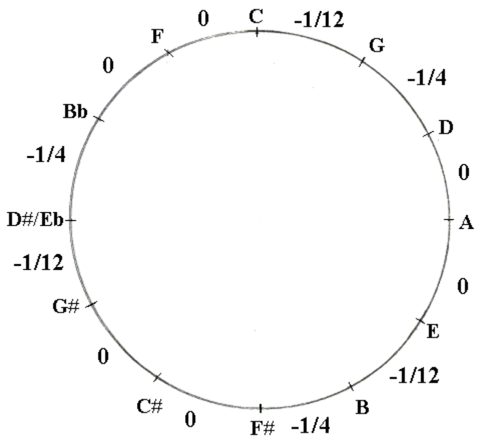

Figure 98 - Neidhardt 1732 - 5th circle IX temperament

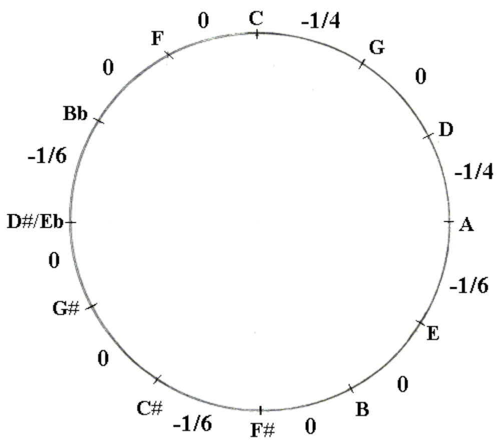

Figure 99 - Neidhardt 1732 - 5th circle X temperament

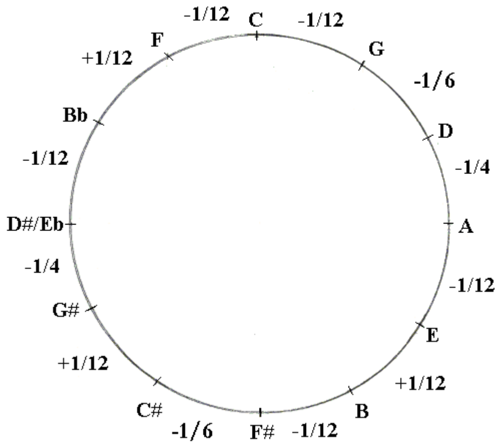

Figure 100 - Neidhardt 1732 - 5th circle XI temperament

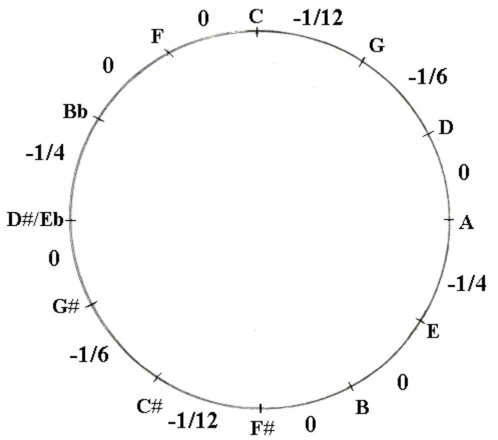

Figure 101 - Neidhardt 1732 - 5th circle XII temperament

1732 – Examples Neidhardt's temperaments:

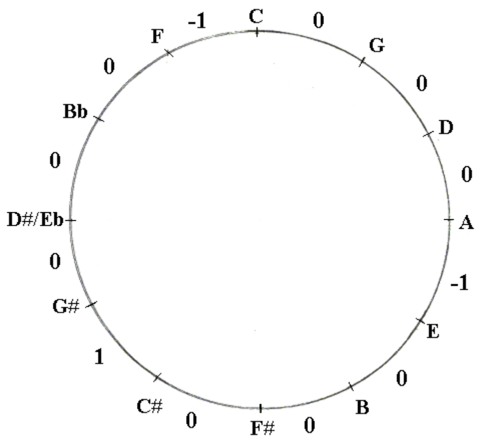

Figure 102 - Neidhardt 1732 - Example I temperament

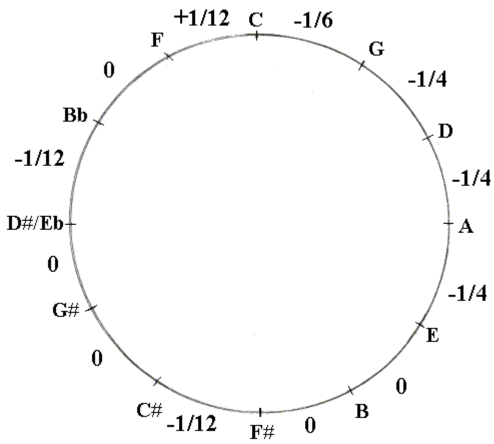

Figure 103 - Neidhardt 1732 - Example II temperament

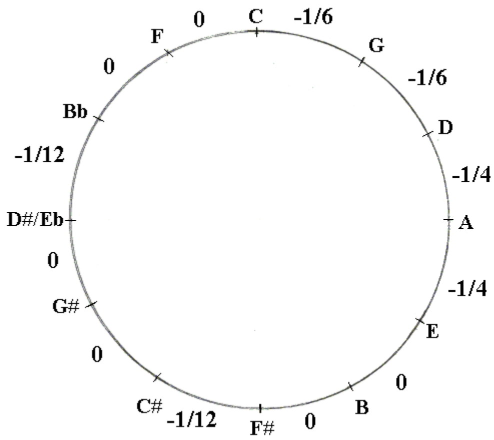

Figure 104 - Neidhardt 1732 - Example III temperament

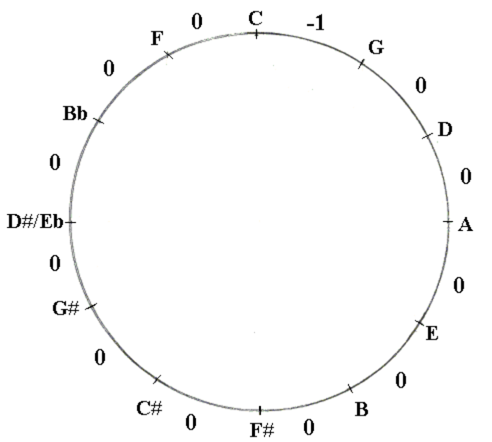

Figure 105 - Neidhardt 1732 - Phytagorean example temperament

Sorge

Georg Andreas Sorge (21 March 1703, Mellenbach - 4 April 1778, Lobenstein) proposed several temperaments, the most important of which he himself calls

Temperamentum inequale modernum (as opposed to Silbermann's temperament which he calls *Temperamentum inequale vetus*). This temperament dates from 1758[327] and is formed by a combination of several just fifths and other fifths reduced by 1/6 and 1/12 of a Pythagorean comma:

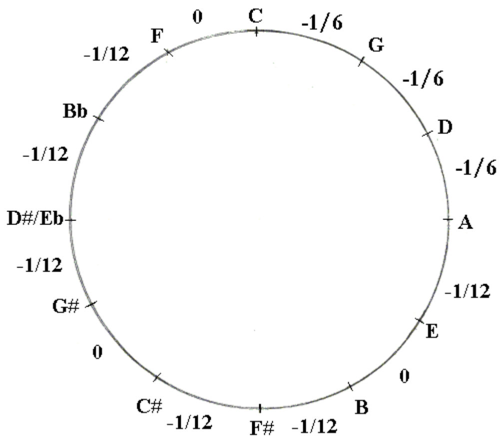

Figure 106 - Sorge 1758 temperament

He proposed other previous temperaments in 1744:[328]

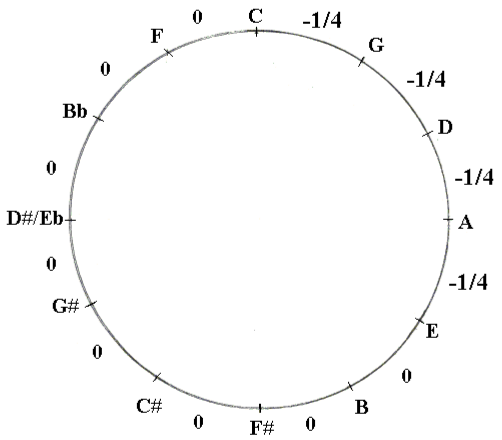

Figure 107 - Sorge 1744 I temperament

[327] SORGE, Georg Andreas: *Zuverlässige Anweisung, Claviere und Orgeln behörig zu temperiren und zu stimmen, nebst einem Kupfer, welches die Ausmessung und Ausrechnung der Temperatur, wie auch das Telemannische Intervallen-System... darstellet; Auf Veranlassung Herrn Barthold Fritzens... herausgegebenen mechanischen Art zu stimmen, und zur Vertheidigung gegen desselben Angrif entworfen*. Lobenstein & Leipzig: Georg Andreas Sorge & G. F. Authenrieth, 1758.

[328] SORGE, Georg Andreas: *Anweisung zur Stimmung und Temperatur sowohl der Orgelwerke, als auch anderer Instrumente, sonderlich aber des Claviers.* Hamburg: Piscator, 1744.

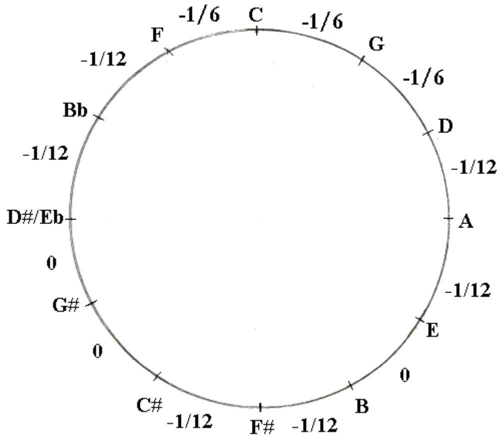

Figure 108 - Sorge 1744 II temperament

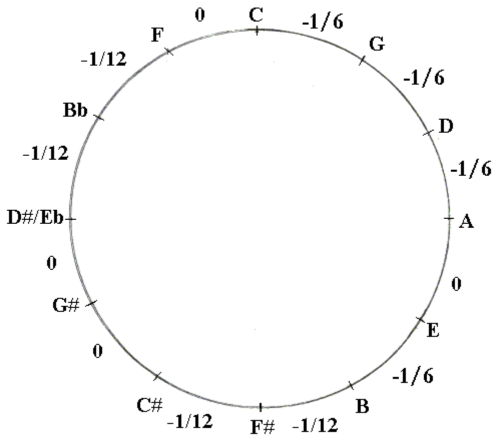

Figure 109 - Sorge 1744 III temperament

Appendix 5: Definition of temperaments by beat rates

The basilar membrane in the inner ear is responsible for the sensation of frequency produced by a periodic wave in the brain. The wave affects a determinate point of the membrane depending on the frequency of the vibration. Thus, if a wave is composed of several pure tones superimposed one on another, several points of the membrane respond. In summary, it can be said that the basilar membrane works as a kind of “biological” spectrum analyzer.

Consequently, when two tones with similar frequency reach the ear, their responses are superimposed on the basilar membrane and it is said that they are in the same critical band. Experimenting with pure sounds, it has been shown that the set of audible frequencies falls into 24 critical bands, more or less. Each one of these covers approximately a length of 1.3mm on the basilar membrane, which is equivalent to approximately 1,300 neurons or nerve cells connected to the auditory nerve.

A central frequency in the auditory range has a critical bandwith which is almost constant for frequencies lower than 500Hz (approximately C4); higher than this the critical bandwith increases almost proportionally to the frequency.

Definition and calculation of beat rates

Starting with the equation of a wave in a one-dimension medium:

$$y(x,t) = A\cos(2\pi ft + \vec{k}\cdot\vec{x})$$

a superimposition of two pure sounds with the same amplitude can be represented as:

$$y(x,t) = A\cos(2\pi f_1 t + \vec{k}\cdot\vec{x}) + A\cos(2\pi f_2 t + \vec{k}\cdot\vec{x})$$

Thus, according to the following trigonometric identity:

$$\cos x + \cos y = 2\cos(\frac{y-x}{2})\cos(\frac{y+x}{2})$$

the superimposition can be rewritten in this way:

$$y(x,t) = 2A\cos(\frac{2\pi(f_2 - f_1)t}{2})\cos(\frac{2\pi(f_2 + f_1)t}{2} + \vec{k}\cdot\vec{x})$$

where the first part of the expression represents a variation in the amplitude of the resulting sound whose frequency corresponds to the difference between both frequencies of the original sounds.

Depending on the magnitude of the frequency difference $\Delta f = f_2 - f_1$, the effect is:

a) *Δf<10Hz*: The variation in amplitude of the envelope is perceived as reinforcements of the sound which can be identified as beats. The maximum of amplitude of the envelope is produced when phases of both sounds coincide.
b) *Δf >15Hz*: Beats disappear since they are too fast and they are perceived as mixed. At this moment a roughness in the sound is heard.
c) Increasing the frequency difference, both sounds can be heard separately after a moment, still with roughness.
d) A moment later, this roughness disappears.

In the range in which beats are produced, their frequency is equal to the difference between the frequencies of the original sound. Consequently the maximum possible number of beats is approximately 10.

This roughness of the sound is identified as dissonance.

Helmholtz observed that the maximum roughness is produced when the difference between the frequencies of the two sounds is equal to 32. Plolmp & Levelt's later experiments have demonstrated that the point of maximum dissonance, in almost all frequency margins, is produced in around 1/4 of the critical bandwith. This criterion coincides with Helmhotz'criterion for frequencies close to 500Hz.

This process describes the dissonance curve for a pair of pure sounds. Generalising this result for more complex tones composed of several partials, it is assumed that the total dissonance is equal to the sum of the dissonance of all pairs of partials which interact with each other.

On the other hand, the calculation of beats is significant when the frequency difference is higher than the critical bandwidth but one of the first harmonics of one of the original sounds has approximately the same frequency as another of the first harmonics of the other original sound.

In a superimposition of harmonic sounds:

$$y(\vec{x},t) = \sum_{n=0}^{N-1} A_n \cos(2\pi f_n t + \vec{k}\cdot\vec{x}) + \sum_{m=0}^{M-1} B_m \cos(2\pi g_m t + \vec{k}\cdot\vec{x})$$

where $f_n = nf_0, 1 \le n \le N-1$ and $g_m = mg_0, 1 \le m \le M-1$, the set of frequency differences is:

$$\{|mg_o - nf_0|, 1 \le n \le N-1, 1 \le m \le M-1\}$$

Generalizing the calculation, the resulting (or the most meaningful) number of beat rates can be considered as the maximum number of the set within the range of production, that is to say, the maximum number of the set which is less than 10.

Properties of beat rates

The following properties of beat rates can be considered in relation to the different operations with intervals:

1) Taking into account that an octave is always pure, that is $f_b = 2f_a$, it does not beat:
   $f_b - 2f_a = 2f_a - 2f_a = 0$
2) Change of octave:
   i. An interval without beats in a given octave is without beats in the octave above:
   $f_b - f_a = 0 \Rightarrow 2f_b - 2f_a = 2(f_b - f_a) = 0$
   The same property can be generalized for any number of octaves:
   $f_b - f_a = 0 \Rightarrow 2^k f_b - 2^k f_a = 2^k (f_b - f_a) = 0$
   ii. An interval which beats at a certain rate beats twice as fast in the octave above and twice as slow in the octave below:
   $2f_b - 2f_a = 2(f_b - f_a)$
   $\frac{f_b}{2} - \frac{f_a}{2} = \frac{f_b - f_a}{2}$
   The same property can be generalized for any number of octaves:
   $2^k f_b - 2^k f_a = 2^k (f_b - f_a)$
3) Beat rates for some intervals:
   i. In a fifth, the third harmonic of the lowest note has to be compared with the second of the highest note, and the interval beats as:
   $b = 3f_b - 2f_a$
   Consequently, a just fifth does not beat:
   $f_b = \frac{3}{2} f_a \Rightarrow 3f_a - 2f_b = 3f_a - 3f_a = 0$
   Changing the octave:
   $\overline{b} = 3(2^k f_b) - 2(2^k f_a) = 2^k (3f_b - 2f_a) = 2^k b$
   ii. In a fourth, the fourth harmonic of the lowest note has to be compared with the third of the highest note, and the intervals beats as:
   $b = 4f_b - 3f_a$
   Consequently, a just fourth does not beat:
   $f_b = \frac{4}{3} f_a \Rightarrow 4f_a - 3f_b = 4f_a - 4f_a = 0$
   Changing the octave:
   $\overline{b} = 4(2^k f_b) - 3(2^k f_a) = 2^k (4f_b - 3f_a) = 2^k b$
4) Inversion of fifths and fourths:
   i. Inverting a fifth (*e.g.* C1-G1) such that its lower note sounds an octave higher (*i.e.* G1-C2 for the given example), yields an interval that beats twice as fast (the fourth harmonic – double octave – of the lowest note has to be compared with the third of the highest):

$$\overline{b} = 3(2f_b) - 4f_a = 6f_b - 4f_a = 2(3f_b - 2f_a) = 2b$$

Inverting a fifth (*e.g.* C2-G2) such that its upper note sounds an octave lower (*i.e.* G1-C2 for the given example), yields an interval which beats at the same rate (the fourth harmonic of the lowest has to be compared with the third of the highest):

$$\overline{b} = 3f_b - 4\left(\frac{f_a}{2}\right) = 3f_b - 2f_a = b$$

ii. Inverting a fourth (*e.g.* C1-F1) such that its lower note sounds an octave higher (*i.e.* F1-C2 for the given example), yields an interval which beats at the same rate (the third harmonic of the lowest note has to be compared with the second of the highest):

$$\overline{b} = 2(2f_b) - 3f_a = 4f_b - 3f_a = b$$

iii. Inverting a fourth (*e.g.* C2-F2) such that its upper note sounds an octave lower (*i.e.* F1-C2 for the given example), yields an interval which beats twice as slow (the third harmonic of the lowest note has to be compared with the second of the higest):

$$\overline{b} = 2f_b - 3\left(\frac{f_a}{2}\right) = 2f_b - \frac{3}{2}f_a = \frac{1}{2}(4f_b - 3f_a) = \frac{b}{2}$$

Definition of temperaments[329]

Temperaments defined by beat rates are also represented by the circle of fifths but the numbers of the graphic represent the beat rates of each interval instead of the fractions of a comma which they are reduced by. According to the last paragraph, the closest harmonics correspond to the fifths $3f_a$ and $2f_b$, where f_a is the lowest frequency and f_b the highest. Thus beat rates are equal to $3f_a - 2f_b$ and this difference is null if the interval is pure. On the other hand, since the octave is pure, its beat rate or the frequency difference between its two frequencies is null.

Considering a scale of 12 notes and defining it by the frequencies of their notes, such that f_0 is the lowest note and f_{11} the highest, and considering the notes of the octave above starting in f_{12}, it is easy to deduce that a fifth will be represented by one interval $f_b - f_a$ such that $b - a = 7$ and, for an octave, such that $b - a = 12$.

If b_n are the beat rates, f_n are the frequencies of the notes of the scale.

Starting with the first note of the scale (*e.g.* C), the beat rate for each fifth must be equal to each b_n, that is:

$$3f_n - 2f_{n+1} = b_n$$

When the fifth is higher than an octave above the first note, an octave has to be reduced, that is:

$$2f_n - f_{n+12} = 0$$

In order to include all notes within the same octave for which beat rates are defined and considering only f_n such that $0 \leq n \leq n_0 - 1, n_0 = 12$ and, for the complete circle, the following system of equations can be developed:

[329] *Vid.* FRANCIS, John Charles: "The Esoteric Keyboard Temperaments…", *op. cit.*

$$
\begin{cases}
3f_0 - 2f_7 = b_0 \\
3f_7 - 2f_{14} = b_1 \\
2f_2 - f_{14} = 0 \\
3f_2 - 2f_9 = b_2 \\
3f_9 - 2f_{16} = b_3 \\
2f_4 - f_{16} = 0 \\
3f_4 - 2f_{11} = b_4 \\
3f_{11} - 2f_{18} = b_5 \\
2f_6 - f_{18} = 0 \\
3f_6 - 2f_{13} = b_6 \\
2f_1 - f_{13} = 0 \\
3f_1 - 2f_8 = b_7 \\
3f_8 - 2f_{15} = b_8 \\
2f_3 - f_{15} = 0 \\
3f_3 - 2f_{10} = b_9 \\
3f_{10} - 2f_{17} = b_{10} \\
2f_5 - f_{17} = 0 \\
3f_5 - 2f_{12} = b_{11} \\
2f_0 - f_{12} = 0
\end{cases}
\quad
\begin{cases}
f_{14} = 2f_2 \\
f_{16} = 2f_4 \\
f_{18} = 2f_6 \\
f_{13} = 2f_1 \\
f_{15} = 2f_3 \\
f_{17} = 2f_5 \\
f_{12} = 2f_0
\end{cases}
\quad
\begin{cases}
3f_0 - 2f_7 = b_0 \\
3f_7 - 4f_2 = b_1 \\
3f_2 - 2f_9 = b_2 \\
3f_9 - 4f_4 = b_3 \\
3f_4 - 2f_{11} = b_4 \\
3f_{11} - 4f_6 = b_5 \\
3f_6 - 4f_1 = b_6 \\
3f_1 - 2f_8 = b_7 \\
3f_8 - 4f_3 = b_8 \\
3f_3 - 2f_{10} = b_9 \\
3f_{10} - 4f_5 = b_{10} \\
3f_5 - 4f_0 = b_{11}
\end{cases}
\quad
\begin{cases}
3f_1 - 2f_2 = b_1 \\
3f_2 - 4f_3 = b_2 \\
3f_3 - 2f_4 = b_3 \\
3f_4 - 4f_5 = b_4 \\
3f_5 - 2f_6 = b_5 \\
3f_6 - 4f_7 = b_6 \\
3f_7 - 4f_8 = b_7 \\
3f_8 - 2f_9 = b_8 \\
3f_9 - 4f_{10} = b_9 \\
3f_{10} - 2f_{11} = b_{10} \\
3f_{11} - 4f_{12} = b_{11} \\
3f_{12} - 4f_0 = b_{12}
\end{cases}
$$

We can generalise this system as follows:

$$
\begin{cases}
3f_i - 2a_i f_{i+1} = b_i,\ a_i \in \{1, 2\},\ i \in \{1, \dots, N\} \\
f_{N+1} = f_1
\end{cases}
$$

The solution of this system yields the complete set of frequencies of the scale:

$$
\begin{cases}
f_1 = \dfrac{\sum_{i=1}^{N} \left(\frac{3}{2}\right)^{N-i} \left(\prod_{j=i}^{N} \dfrac{1}{a_j} \right) b_i}{\left(\frac{3}{2}\right)^{N} \prod_{i=1}^{N} \dfrac{1}{a_i} - 1} \\
f_n = \dfrac{1}{2a_{n-1}} \left(3f_{n-1} - b_{n-1} \right),\ 2 \le n \le N
\end{cases}
$$

After this, to get the ratios of the scale it is only necessary to calculate them by dividing the frequencies obtained, that is:

$$
r_n = \frac{f_{n+Q \bmod N}}{f_n},\ 0 \le n \le N-1,\ N = 12,\ Q = 7
$$

$$
n + Q \bmod N = n + Q - E\left\lfloor \frac{n+Q}{N} \right\rfloor N
$$

To get the scale starting on a particular note, *e.g.* C, the corresponding fifth can be considered as the first in the circle of fifths in which beat rates were defined.

The following scheme is very clear and illustrative of the process:[330]

[330] *Vid.* FRANCIS, John Charles: "The Esoteric Keyboard Temperaments…", *op. cit.*

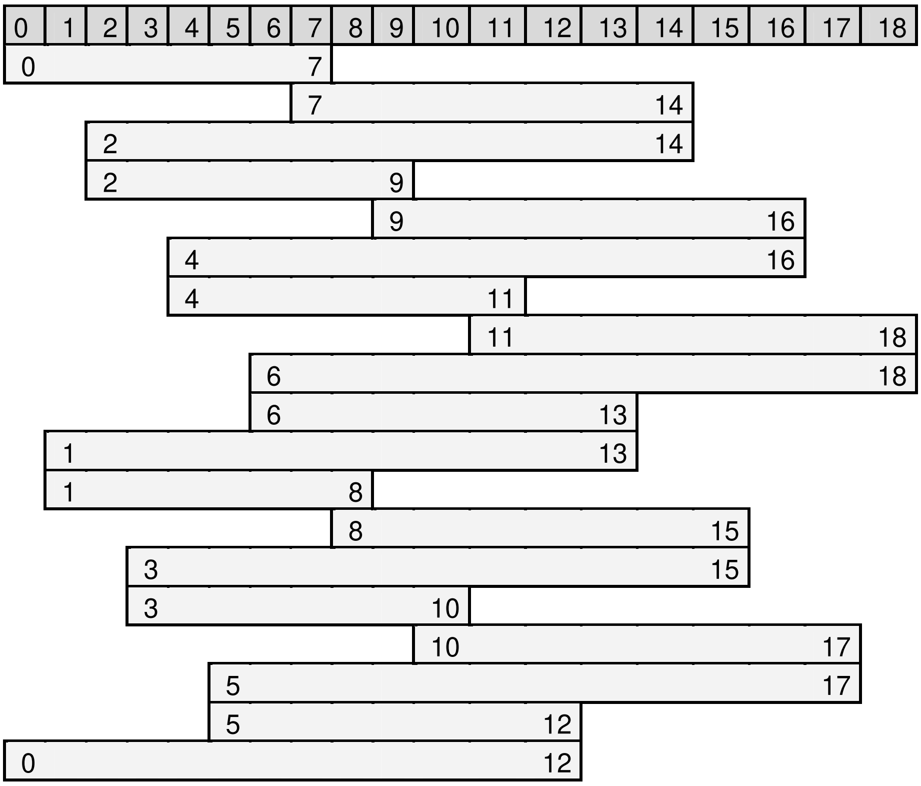

Figure 110 - The procedure used for tuning contiguous semitones using a sequence of fifths on the circle-of-fifths with octave leaps.

Approximation by fractions of a comma

As the Pythagorean comma must be distributed over the circle of fifths, the following relation between the values of beat rates and the fractions of the comma can be established:

$$\alpha(n) = \frac{b_n}{\sum_{i=1}^{N} b_i}$$

where it is easy to check that:

$$\sum_{i=1}^{N} \alpha(n) = \frac{\sum_{i=1}^{N} b_i}{\sum_{i=1}^{N} b_i} = 1$$

This means that each tempering unit theoretically corresponds to a fraction of a Pythagorean comma given by the previous expression. The tempering value of each fifth of the circle can be worked out as follows:

$$\Delta q(n) = -\frac{b_n}{\sum_{i=1}^{N} b_i} \xi_0$$

Starting at this point, the temperament can be treated as a normal ‘good’ temperament based on the Pythagorean intonation.

Appendix 6: Definition of temperaments according to the deviation of thirds

Some temperaments are defined only by the deviation of their thirds and they are usually expressed in *schismata*. In order to calculate the deviation of the fifths and apply the usual procedures, the following operations are necessary:

For each third, the following condition has to be satisfied:

$$\sum_{i=n}^{n+4} q_i + t_n = 11, 1 \leq n \leq N$$

where t_i are the deviations of thirds, q_i the deviations of fifths and $N = n_0$.

Then a system of equations of this form must be solved in order to calculate the deviations of the fifths:

$$\sum_{i=n}^{n+4} q_i = 11 - t_n, 1 \leq n \leq N$$

taking into account that $q_{N+n} = q_n$.

Moreover, the following properties must also be satisfied:

1) $\sum_{i=1}^{N} q_i = 12$

2) $\sum_{i=1}^{N/4} t_{n+4i} = 21, 1 \leq n \leq 4$

After this, intervals of the scale can be calculated according to the procedure indicated in 'Appendix 3: Representation of 'good' temperaments' (section 'Representation with *schismata*').

Appendix 7: Tuning circumstances at Bach's time

Several pitch standards were in use in Bach's day. The best known standards are:

- *Cornet-ton* pitch, where A=460-470 Hz. Its average is A=465 Hz. This pitch was used in most German organs made in Bach's lifetime.
- *Cammerton (Kammerton)* pitch, which has been standarized today at A=415 Hz (a tone lower than *Cornet-ton*). This pitch was used in Bach's chamber and orchestral works and it has been standardised today for the benefit of period-instrument performers.
- *Tief-Cammerton* pitch, which was one semitone lower.

These pitches could be used simultaneously and this fact created the necessity for coordinating the intonation of the respective instruments.

Appendix 8: Dissonance theory

Mathematical model for spectra

If we consider that a spectrum is composed of a series of partial tones, harmonic or not, its mathematical model consists of a Dirac delta train (a "pulse train" of Dirac delta measures or Dirac comb) and, at the same time, is equivalent to the Fourier transform of a wave composed of the sum of several sine functions:

$$X(f) = \sum_{i=0}^{N-1} A_i \delta(f - f_i)$$

If we identify this model as a set of spectral peaks, we can represent it as follows:

$$E = \{f_i, A_i, 0 \le i < N\}$$

where N is the number of spectral peaks that make up the spectrum, f_i the frequencies of each partial tone and A_i, its related amplitudes. If these frequencies are integer multiples of another fundamental frequency f_0 (which may exist or not), the spectrum is harmonic, i.e., if $f_i = (i+1)f_0$.

Calculating Dissonance Curves[331]

Reinier Plomp & Willem J. M. Levelt curves can be conveniently parameterized by a model of the form:

$$d(x) = e^{-b_1 x} - e^{-b_2 x}$$

where x represents the difference in frequency between two sinusoids, and b_1 and b_2 determine the rates at which the function rises and falls. Using a gradient minimization of the squared error algorithm between the averaged data and the curve $d(x)$ gives values of: $b_1 = 3.5$ and $b_2 = 5.75$.

The dissonance function $d(x)$, on the other hand, can be scaled so that the curves for different base frequencies and with different amplitudes are represented conveniently. After this transformation, the dissonance between sinusoids at frequencies f_1 and f_2 (for $f_1 < f_2$) and with amplitudes a_1 and a_2 can be expressed as follows:

$$d(f_1, f_2, a_1, a_2) = a_{12}\left[e^{-b_1 s(f_2 - f_1)} - e^{-b_2 s(f_2 - f_1)}\right]$$

where

$$s = \frac{x^*}{s_1 f_1 + s_2}$$

and

$$a_{12} = a_1 a_2$$

The variable x^* represents the point of maximum dissonance which has been obtained from the derivative function. The value $x^* = 0.24$ is obtained taking the chosen values for constants b_1 and b_2.

The s parameters allow us to interpolate between the curves s at different frequencies by sliding them along the frequency axis so that it begins at f_1, and by stretching (compressing) it so that the maximum dissonance occurs at the appropriate frequency. Making a squared error minimization algorithm again, the values $s_1=0.021$ and $s_2=19$ are determined.

The parameter a_{12}, on the other hand, allows components with smaller amplitude to contribute less to the total dissonance measure than those components with a larger amplitude.

In general, if a complex sound F with n partial tones with frequencies $f_1<f_2<\ldots<f_n$ and with amplitudes a_i for $i=1,2,\ldots,n$ is considered, its dissonance can be calculated as the sum of the dissonances of all pairs of partials:

[331] This mathematical development is given in SETHARES, William A.: "Local consonance...", *op. cit.*, and SETHARES, William A.: *Tuning..., op. cit.*, appendix E, pp. 299-302.

$$D_F = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(f_i, f_j, a_i, a_j)$$

obtaining the expression of the intrinsic or inherent dissonance of the sound F.

If two notes with timbre F are played simultaneously at an interval with ratio α, the resulting sound has a dissonance that is the same as that of a sound with frequencies f_i and αf_i and amplitudes a_i, for $i=1,2,...,n$. Thus, considering the previous sound F and other sound αF which contains a spectrum with frequencies αf_1, αf_2, ..., αf_n (with the same amplitudes), the dissonance of F at an interval α is given by the following expression:

$$D_F(\alpha) = D_F + D_{\alpha F} + \sum_{i=1}^{n}\sum_{j=1}^{n} d(f_i, \alpha f_j, a_i, a_j)$$

where $D_F(\alpha)$ is the dissonance curve generated by F over all intervals of interest α.[332]

For the case where an interval is formed by sounds with different spectrums for each note, the dissonance curve can be worked out in this way:

$$D_{F,G}(\alpha) = D_F + D_{\alpha G} + \sum_{i=1}^{n}\sum_{j=1}^{n} d(f_i, \alpha g_j, a_i, b_j)$$

where F and G are the spectrums and a_i and b_i their respective amplitudes.

Related spectra and scales

A spectrum and a scale are said to be related if the dissonance curve for that spectrum has minima at scale positions.[333] These minima correspond to points of maximum consonance and they constitute good candidates to be intervals of a suitable scale to work with an instrument with that spectrum.

On the one hand, starting with a given instrument, a suitable scale can be found according to the dissonance curve of its spectrum. The spectrum must be modeled by a set of peaks as explained in the previous section. The points for which a minimum of the dissonance is yielded will constitute the intervals of the scale, at least some of them. Depending on the number of peaks of the spectral model, more or fewer intervals of the scale can be obtained. Likewise, these intervals will have different levels of dissonance and a classification of them according to their sensory dissonance can be proposed. Procedures and theories to obtain more or fewer intervals according to the number of peaks of the spectral models can be developed but this question is beyond our scope in this paper.

On the other hand, starting with a desired scale, a concrete spectrum can be found so that its dissonance curve has minima in the points corresponding to the intervals of the scale.[334]

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[332] An algorithm to calculate and draw the dissonance curve which is programmed with MATLAB and BASIC is given in SETHARES, William A.: *Tuning..., op. cit.*, appendix E, pp. 299-302. My own implementation of the algorithm in the programme *SpecMusic* has been programmed with C++ language.

[333] *Vid.* SETHARES, William A.: *Tuning..., op. cit.*, chapter 9, p. 89.

[334] A systematic procedure has been developed by Sethares. *Vid.* SETHARES, William A.: *Tuning..., op. cit.*, chapter 10, pp. 211-233.

Calculating Dissonance Scores and Total Dissonance[335]

According to the expression given for sensorial dissonance D_F of an interval with fundamental frequencies f_i and f_j, the total dissonance *TD* for a musical fragment with *m* notes is defined as the sum of the dissonances due to the intervals formed by each pair of notes being combined with each other. Moreover, the dissonances are weighted by the period of time during which this pair of notes sounds simultaneously, that is:

$$TD = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} D_F(f_i / f_j) t(i,j)$$

where *m* is the number of notes of the scale, *t (i, j)* is the total time during which the notes *i* and *j* sound simultaneously, and the term $D_F(f_j / f_i)$ corresponds to the dissonance of the interval between them.[336] This interval can be obtained from the expression given in the previous section and determines the dependency of the total dissonance with the tuning.

On the other hand, the effect of the decrease of the amplitude has to take into account particularly in keyboard instruments. In these instruments, the amplitude of a single held note decreases with time considerably but, at the same time, it increases significantly each time a succeeding note is played due to the effect of coupling or resonance. Thus, in musical fragments fast enough, such a distribution is a reasonable approximation.

It can be shown that the previous sum is equivalent to the sum of the dissonances of each chord of the score, taking their duration as a temporal value.

Another important observation is the very small variation between different values of the total dissonance taking different tunings (less than one per cent between usual tunings[337]). Because of this, it is necessary to look for other expressions for the calculation of the total dissonance. The following solutions can be considered:

1) Expressing the result in parts per thousand of the difference between the obtained measure for the considered tuning and the obtained measure for Equal Temperament:[338]

   $$TD' = 1000\,(TD - TD_0)$$

   The difference of one part per thousand in typical musical contexts is clearly audible to a trained musical ear.

2) Applying logarithms and substituting the previous subtraction by a division (likewise in parts per thousand):[339]

   $$TD' = 1000 \log\left(\frac{TD}{TD_0}\right)$$

   This measure provides a larger variation between different results and allows us to obtain conclusions more easily.

---

335 *Vid.* SETHARES, William A.: *Tuning..., op. cit.*, chapter 9, pp. 189-210.

336 The sum corresponds to all possible combinations of the *m* notes.

337 For this reason, a numerical precision of 9 decimal places or greater is advisable for the calculations of the Total Dissonance.

338 This is the solution suggested by Sethares. On the other hand, one has to take into account that the number of notes of the scale in which the composition is based must be the same as the number of notes of the Equal Temperament. This number is 12 for the majority of the musical examples in Western music but it can be different in other kinds of music.

339 This is the solution I have suggested in MARTÍNEZ RUIZ, Sergio: *Desarrollo de un software..., op. cit.*

In the previous equations, *TD* indicates the total dissonance obtained by the given expression; *TD'* indicates the modified expression; and TD_0 refers to the total dissonance worked out for Equal Temperament. Thus TD_0 constitutes a reference value in relation to which the calculation of the dissonance will be expressed for the rest of tunings and temperaments. Consequently, for Equal Temperament a null value will be obtained for the dissonance. In the present paper the second expression has been taken for the calculation of the total dissonance.

Appendix 9: Tables

In the pages below, the following tables are shown:

- **Table 1: Definition of temperaments:** Comparative table for all temperaments given in this work (expressed as the units of definition).
- **Table 2: Intervals of temperaments:** Values for all temperaments (expressed as ratios) given in this work.
- **Table 3: Results obtained:** Total Dissonance for all pieces of the *Well-Tempered Clavier* studied and for all temperaments given in this work.
- **Table 4: Key signatures in Bach's clavier and organ works:** Frequency of the occurrence of sharps / flats in Bach's clavier and organ works derived from BWV (expressed as number of movements).
- **Table 5: Correlations** between Bach temperaments and the frequency of the occurrence of sharps / flats in Bach's clavier and organ Works.

Table 1: Definition of temperaments

HISTORICAL TEMPERAMENTS	UNIT	Eb-Bb	Bb-F	F-C	C-G	G-D	D-A	A-E	E-B	B-F#	F#-C#	C#-G#	G#-D#
Kirnberger I	SC	0	0	0	0	0	-1	0	0	0	-1/11	0	0
Kirnberger II	SC	0	0	0	0	0	-1/2	-1/2	0	0	-1/11	0	0
Kirnberger III	SC	0	0	0	-1/4	-1/4	-1/4	-1/4	0	0	-1/11	0	0
Werckmeister III	PC	0	0	0	-1/4	-1/4	-1/4	0	0	-1/4	0	0	0
Werckmeister IV	PC	+1/3	-1/3	0	-1/3	0	-1/3	0	-1/3	0	-1/3	0	+1/3
Werckmeister V	PC	0	0	-1/4	0	0	-1/4	-1/4	0	0	-1/4	-1/4	+1/4
Vallotti	SC	0	-1/11	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	0
Tartini - Vallotti	PC	0	0	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	0
Neidhardt 1724 I - Village = Neidhardt 1732 - 3rd circle II - Small city	PC	-1/12	0	0	-1/6	-1/6	-1/6	-1/6	-1/12	-1/12	0	0	-1/12
Neidhardt 1724 II - Small city = Neidhardt 1732 - 5th circle VIII - Big city	PC	0	-1/12	-1/12	-1/6	-1/6	-1/6	-1/12	0	-1/12	-1/12	-1/12	0
Neidhardt 1724 III - Big city	PC	-1/12	-1/12	0	-1/6	-1/6	-1/6	-1/12	0	-1/12	-1/12	-1/12	0
Neidhardt 1724 IV - Court = Equal Temperament	PC	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12
Neidhardt 1732 - 5th circle I - Court = Equal Temperament	PC	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12
Neidhardt 1732 - 5th circle II	PC	+1/6	-1/6	-1/6	-1/6	+1/6	-1/6	-1/6	-1/6	+1/6	-1/6	-1/6	-1/6
Neidhardt 1732 - 5th circle III	PC	-1/6	0	-1/6	0	-1/6	0	-1/6	0	-1/6	0	-1/6	0
Neidhardt 1732 - 5th circle IV	PC	-1/4	0	0	-1/4	0	0	-1/4	0	0	-1/4	0	0
Neidhardt 1732 - 5th circle V	PC	-1/12	-1/12	-1/6	-1/12	-1/12	-1/6	+1/12	-1/12	-1/6	-1/12	-1/12	0
Neidhardt 1732 - 5th circle VI	PC	-1/12	-1/4	+1/12	-1/12	-1/4	+1/12	-1/12	-1/4	+1/12	-1/12	-1/4	+1/12
Neidhardt 1732 - 5th circle VII	PC	-1/6	-1/6	+1/6	-1/4	-1/6	-1/6	+1/6	-1/6	-1/6	-1/6	+1/6	-1/12
Neidhardt 1732 - 5th circle VIII - Big city = Neidhardt 1724 II - Small city	PC	0	-1/12	-1/12	-1/6	-1/6	-1/6	-1/12	0	-1/12	-1/12	-1/12	0
Neidhardt 1732 - 5th circle IX	PC	-1/4	0	0	-1/12	-1/4	0	0	-1/12	-1/4	0	0	-1/12
Neidhardt 1732 - 5th circle X	PC	-1/6	0	0	-1/4	0	-1/4	-1/6	0	0	-1/6	0	0
Neidhardt 1732 - 5th circle XI	PC	-1/12	+1/12	-1/12	-1/12	-1/6	-1/4	-1/12	+1/12	-1/12	-1/6	+1/12	-1/4
Neidhardt 1732 - 5th circle XII	PC	-1/4	0	0	-1/12	-1/6	0	-1/4	0	0	-1/12	-1/6	0
Neidhardt 1732 - 3rd circle I - Village	PC	0	-1/12	0	-1/12	-1/6	-1/4	-1/4	0	-1/12	0	-1/12	0
Neidhardt 1732 - 3rd circle II - Small city = Neidhardt 1724 I - Village	PC	-1/12	0	0	-1/6	-1/6	-1/6	-1/6	-1/12	-1/12	0	0	-1/12
Neidhardt 1732 - 3rd circle III	PC	+1/6	-1/6	-1/12	-1/6	-1/6	-1/12	-1/6	-1/6	+1/6	-1/6	-1/12	-1/12
Neidhardt 1732 - 3rd circle IV	PC	+1/12	-1/6	0	-1/6	-1/6	-1/6	0	-1/6	0	-1/12	-1/12	-1/12
Neidhardt 1732 - Example I	PC	0	0	-1	0	0	0	-1	0	0	0	1	0

Neidhardt 1732 - Example II	PC	-1/12	0	+1/12	-1/6	-1/4	-1/4	-1/4	0	0	-1/12	0	0
Neidhardt 1732 - Example III	PC	-1/12	0	0	-1/6	-1/6	-1/4	-1/4	0	0	-1/12	0	0
Neidhardt 1732 - Phytagorean example	PC	0	0	0	-1	0	0	0	0	0	0	0	0
Sorge 1758	PC	-1/12	-1/12	0	-1/6	-1/6	-1/6	-1/12	0	-1/12	-1/12	0	-1/12
Sorge 1744 I	PC	0	0	0	-1/4	-1/4	-1/4	-1/4	0	0	0	0	0
Sorge 1744 II	PC	-1/12	-1/12	-1/6	-1/6	-1/6	-1/12	-1/12	0	-1/12	-1/12	0	0
Sorge 1744 III	PC	-1/12	-1/12	0	-1/6	-1/6	-1/6	0	-1/6	-1/12	-1/12	0	0

BACH TEMPERAMENTS	DIR.	START. NOTE	ORDER	UNIT	Eb-Bb	Bb-F	F-C	C-G	G-D	D-A	A-E	E-B	B-F#	F#-C#	C#-G#	G#-D#
Kelletat				PC	0	0	-1/12	-1/4	-1/4	-1/4	-1/6	0	0	0	0	0
Kellner				PC	0	0	0	-1/5	-1/5	-1/5	-1/5	0	-1/5	0	0	0
Barnes				PC	0	0	-1/6	-1/6	-1/6	-1/6	-1/6	0	-1/6	0	0	0
Billeter I				SC	0	0	-1/11	-1/12	-1/12	-1/3	-1/3	-1/12	-1/12	0	0	0
Lindley I				Sch	-0,03	-0,03	1,27	1,93								
Lindley II – Average Neidhardt				PC	-1/24	-1/24	-1/24	-1/6	-1/6	-1/6	-1/8	-1/24	-1/12	-1/24	-1/24	-1/24
Sparschuh	L-R	A	Fifths	BR	0,5	0,5	0,5	0,5	0,5	0	1	1	1	0	0	0
Jira I				PC	0	0	0	-1/6	-1/6	-1/6	-1/6	0	-1/6	-1/4	0	+1/12
Jira II				PC	-1/12	0	0	-1/6	-1/6	-1/6	-1/6	0	-1/12	-1/12	-1/12	0
Zapf	L-R	C	Fifths	BR	0.5	0.5	0.9	1	1	1	0	0	0	0.5	0.5	0.5
Briggs	L-R	C	Fifths	BR	0.5	0.5	1	1	1	1	0	0	0	0.5	0.5	0.5
Francis I				SC	0	0	-14/33	0	0	0	-1/3	-1/3	0	0	+1/3	-1/3
Lehman I	R-L	F	Fifths	SC	-1/12	0	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/12	-1/11
Lehman II	R-L	F	Fifths	PC	-1/12	+1/12	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/12	-1/12
Francis II - Cammerton	L-R	C	Fifths	BR	1	1	1	1	2	2	2	2	2	0	0	0
Francis III - Cornet-ton	R-L	F	Fifths	BR	1	1	2	2	2	2	2	0	0	0	1	1
Dent I	L-R	B	Fifths	PC (SC)	0	0	-1/6	-1/6	-1/6	-1/6	-1/12	-1/12	-1/12	-1/12	-1/12	0
Dent II	L-R	E	Fifths	PC (SC)	0	0	-1/6	-1/6	-1/6	-1/6	-1/6	0	-1/12	-1/12	-1/12	0
Dent III	R-L	A	Fifths	PC (SC)	0	0	-1/6	-1/6	-1/6	-1/6	-1/12	-1/12	-1/12	-1/12	-1/12	0
Ponsford I	L-R	G	Fifths	PC	-1/12	-1/12	-1/12	-1/12	-1/6	-1/6	-1/6	0	0	0	-1/12	-1/12
Ponsford II	L-R	G	Fifths	PC	-1/6	-1/6	-1/6	+1/12	-1/12	-1/12	-1/12	0	0	0	-1/6	-1/6
Maunder I	R-L	F	Fifths	PC	-1/18	0	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/18	-1/18
Maunder II	R-L	F	Fifths	PC	-1/24	-1/24	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/24	-1/24
Maunder III	R-L	F	Fifths	PC	-1/14	-1/14	-1/7	-1/7	-1/7	-1/7	-1/7	0	0	0	-1/14	-1/14
Mobbs-Mackenzie	L-R	C	Fifths	SC	-19/330	-19/330	-37/660	-37/660	-37/660	-37/660	-1/4	-1/4	-1/4	0	0	0
Lucktenberg	R-L	C	Fifths	PC	-1/8	0	-1/8	-1/8	-1/8	-1/8	-1/8	-1/8	0	0	-1/8	0
Jencka	R-L	F	Fifths	PC	-1/18	0	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/18	-1/18
Lehman III	L-R	C	Chrom.	PC	0	-1/6	0	-1/6	-1/6	-1/6	-1/6	0	0	-1/6	+1/6	-1/6
Lindley-Ortgies I	R-L	F	Fifths	PC	-1/24	0	-1/8	-1/8	-1/8	-1/8	-1/8	-1/12	-1/12	-1/12	-1/24	-1/24

Lindley-Ortgies II	R-L	C	Fifths	PC	-1/12	-1/12	+1/12	-1/6	-1/6	-1/6	-1/6	-1/6	0	0	0	-1/12
O'Donnell	L-R	C	Chrom.	PC	-1/12	0	0	-1/6	-1/6	-1/6	0	-1/6	0	-1/12	-1/12	-1/12
Spányi				SC	-1/66	-1/66	-1/66	0	0	-281/660	-281/660	0	0	-109/660	-1/66	-1/66
Interbartolo-Venturino I	R-L	F	Fifths	SC	+7/132	0	-1/4	-1/4	-1/4	-1/4	-1/4	0	0	0	+7/132	+7/132
Interbartolo-Venturino II	R-L	F	Fifths	PC	+1/12	0	-1/4	-1/4	-1/4	-1/4	-1/4	0	0	0	+1/12	+1/12
Interbartolo-Venturino III	R-L	F	Fifths	SC	+1/12	0	-1/4	-1/4	-1/4	-1/4	-1/4	0	0	-1/11	+1/12	+1/12
Billeter IIa				Sch	0	0	0	2,25								
Billeter IIb				Sch	1	0	0	2								
Billeter IIIa	R-L	F	Fifths	Sch	0,5	0	1,75	2								
Billeter IIIb	R-L	C	Fifths	Sch	0,5	0,5	0	1,75								
Jobin	R-L	C	Fifths	SC	+7/132	+7/132	0	-1/4	-1/4	-1/4	-1/4	-1/4	0	0	0	+7/132
Di Veroli I				PC	0	-1/12	-1/6	-1/6	-1/6	-1/12	-1/6	-1/12	-1/12	0	0	0
Di Veroli II				PC	0	-1/12	-1/6	-1/6	-1/6	-1/12	-1/6	-1/12	-1/12	0	+1/12	-1/12
Glueck	R-L	F	Fifths	PC	-1/12	0	-1/8	-1/8	-1/8	-1/8	-1/8	-1/24	-1/24	-1/24	-1/12	-1/12

Table 2: Intervals of temperaments

Kirnberger I		Kirnberger II		Kirnberger III	
C	1,00000	C	1,00000	C	1,00000
C#	1,05350	C#	1,05350	C#	1,05350
D	1,12500	D	1,12500	D	1,11803
Eb/D#	1,18518	Eb/D#	1,18518	Eb/D#	1,18518
E	1,25000	E	1,25000	E	1,25000
F	1,33333	F	1,33333	F	1,33333
F#	1,40625	F#	1,40625	F#	1,40625
G	1,50000	G	1,50000	G	1,49535
G#	1,58025	G#	1,58025	G#	1,58025
A	1,66667	A	1,67705	A	1,67185
Bb	1,77778	Bb	1,77778	Bb	1,77778
B	1,87500	B	1,87500	B	1,87500
C	2,00000	C	2,00000	C	2,00000
Werckmeister III		**Werckmeister IV**		**Werckmeister V**	
C	1,00000	C	1,00000	C	1,00000
C#	1,05350	C#	1,04875	C#	1,05707
D	1,11740	D	1,11993	D	1,12500
Eb/D#	1,18519	Eb/D#	1,18519	Eb/D#	1,18921
E	1,25283	E	1,25424	E	1,25708
F	1,33333	F	1,33333	F	1,33786
F#	1,40466	F#	1,40466	F#	1,41421
G	1,49493	G	1,49324	G	1,50000
G#	1,58025	G#	1,57313	G#	1,58025
A	1,67044	A	1,67232	A	1,68179
Bb	1,77778	Bb	1,78583	Bb	1,78381
B	1,87924	B	1,87289	B	1,88562
C	2,00000	C	2,00000	C	2,00000
Vallotti		**Vallotti - Tartini**			
C	1,00000	C	1,00000		
C#	1,05687	C#	1,05588		
D	1,12035	D	1,11993		
Eb/D#	1,18898	Eb/D#	1,18786		
E	1,25519	E	1,25424		
F	1,33610	F	1,33635		
F#	1,40916	F#	1,40784		
G	1,49690	G	1,49662		
G#	1,58531	G#	1,58382		
A	1,67705	A	1,67610		
Bb	1,78347	Bb	1,78180		
B	1,87889	B	1,87712		
C	2,00000	C	2,00000		

Neidhardt 1724 I – Village		Neidhardt 1724 II – Small city		Neidhardt 1724 III – Big city		Neidhardt 1724 IV – Court (ET)	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05588	C#	1,05707	C#	1,05707	C#	1,05946
D	1,11993	D	1,11993	D	1,11993	D	1,12246
Eb/D#	1,18652	Eb/D#	1,18786	Eb/D#	1,18786	Eb/D#	1,18921
E	1,25424	E	1,25566	E	1,25566	E	1,25992
F	1,33333	F	1,33484	F	1,33333	F	1,33484
F#	1,40784	F#	1,41102	F#	1,41102	F#	1,41421
G	1,49662	G	1,49662	G	1,49662	G	1,49831
G#	1,58382	G#	1,58382	G#	1,58382	G#	1,58740
A	1,67610	A	1,67610	A	1,67610	A	1,68179
Bb	1,77778	Bb	1,78180	Bb	1,77979	Bb	1,78180
B	1,87924	B	1,88349	B	1,88349	B	1,88775
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Neidhardt 1732 – 3rd circle I - Village		Neidhardt 1732 – 3rd circle II - Small city		Neidhardt 1732 – 3rd circle III		Neidhardt 1732 – 3rd circle IV		Neidhardt 1732 – 3rd circle V	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05588	C#	1,05588	C#	1,05707	C#	1,05707	C#	1,05946
D	1,12120	D	1,11993	D	1,11993	D	1,11993	D	1,12246
Eb/D#	1,18652	Eb/D#	1,18652	Eb/D#	1,18652	Eb/D#	1,18652	Eb/D#	1,18921
E	1,25283	E	1,25424	E	1,25566	E	1,25708	E	1,25850
F	1,33330	F	1,33333	F	1,33484	F	1,33333	F	1,33635
F#	1,40784	F#	1,40784	F#	1,41262	F#	1,41102	F#	1,41262
G	1,49831	G	1,49662	G	1,49662	G	1,49662	G	1,49831
G#	1,58203	G#	1,58382	G#	1,58382	G#	1,58382	G#	1,58740
A	1,67610	A	1,67610	A	1,67800	A	1,67610	A	1,68179
Bb	1,77979	Bb	1,77778	Bb	1,78381	Bb	1,78180	Bb	1,78180
B	1,87924	B	1,87924	B	1,87924	B	1,88136	B	1,88562
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000

Neidhardt 1732 – 5th circle I - Court (ET)		Neidhardt 1732 – 5th circle II		Neidhardt 1732 – 5th circle III		Neidhardt 1732 – 5th circle IV	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05946	C#	1,06066	C#	1,06066	C#	1,05707
D	1,12246	D	1,12500	D	1,12246	D	1,12120
Eb/D#	1,18921	Eb/D#	1,18786	Eb/D#	1,19055	Eb/D#	1,18921
E	1,25992	E	1,25992	E	1,25992	E	1,25708
F	1,33484	F	1,33635	F	1,33635	F	1,33333
F#	1,41421	F#	1,41741	F#	1,41421	F#	1,41421
G	1,49831	G	1,49662	G	1,50000	G	1,49493
G#	1,58740	G#	1,58740	G#	1,58740	G#	1,58561
A	1,68179	A	1,68369	A	1,68369	A	1,68179
Bb	1,78180	Bb	1,78583	Bb	1,78180	Bb	1,77778
B	1,88775	B	1,88562	B	1,88988	B	1,88562
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Neidhardt 1732 – 5th circle V		Neidhardt 1732 – 5th circle VI		Neidhardt 1732 – 5th circle VII		Neidhardt 1732 – 5th circle VIII - Big city	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05946	C#	1,05946	C#	1,05469	C#	1,05707
D	1,12246	D	1,11993	D	1,11867	D	1,11993
Eb/D#	1,19055	Eb/D#	1,18921	Eb/D#	1,18786	Eb/D#	1,18786
E	1,26134	E	1,25992	E	1,25850	E	1,25566
F	1,33635	F	1,33183	F	1,33033	F	1,33484
F#	1,41421	F#	1,41421	F#	1,40943	F#	1,41102
G	1,49831	G	1,49831	G	1,49493	G	1,49662
G#	1,58740	G#	1,58382	G#	1,58561	G#	1,58382
A	1,67989	A	1,68179	A	1,67421	A	1,67610
Bb	1,78381	Bb	1,78180	Bb	1,77778	Bb	1,78180
B	1,88988	B	1,88349	B	1,88349	B	1,88349
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Neidhardt 1732 – 5th circle IX		Neidhardt 1732 – 5th circle X		Neidhardt 1732 – 5th circle XI		Neidhardt 1732 – 5th circle XII	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05827	C#	1,05588	C#	1,05707	C#	1,05946
D	1,11993	D	1,12120	D	1,12120	D	1,12120
Eb/D#	1,18921	Eb/D#	1,18786	Eb/D#	1,18652	Eb/D#	1,18921
E	1,25992	E	1,25424	E	1,25566	E	1,25708
F	1,33333	F	1,33333	F	1,33484	F	1,33333
F#	1,41102	F#	1,41102	F#	1,41262	F#	1,41421
G	1,49831	G	1,49493	G	1,49831	G	1,49831
G#	1,58740	G#	1,58382	G#	1,58740	G#	1,58561
A	1,67989	A	1,67610	A	1,67610	A	1,68179
Bb	1,77778	Bb	1,77778	Bb	1,77778	Bb	1,77778
B	1,88775	B	1,88136	B	1,88562	B	1,88562
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Neidhardt 1732 – Example I		Neidhardt 1732 – Example II		Neidhardt 1732 – Example III		Neidhardt 1732 – Phytagorean example	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05350	C#	1,05350	C#	1,05469	C#	1,05350
D	1,12500	D	1,11867	D	1,11993	D	1,10986
Eb/D#	1,20135	Eb/D#	1,18519	Eb/D#	1,18652	Eb/D#	1,18519
E	1,24859	E	1,25000	E	1,25141	E	1,24859
F	1,35152	F	1,33183	F	1,33333	F	1,33333
F#	1,40466	F#	1,40625	F#	1,40784	F#	1,40466
G	1,50000	G	1,49662	G	1,49662	G	1,47981
G#	1,60181	G#	1,58025	G#	1,58203	G#	1,58025
A	1,68750	A	1,67232	A	1,67421	A	1,66479
Bb	1,80203	Bb	1,77577	Bb	1,77778	Bb	1,77778
B	1,87289	B	1,87500	B	1,87712	B	1,87289
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Sorge 1758		Sorge 1744 I		Sorge 1744 II		Sorge 1744 III	
C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05707	C#	1,05350	C#	1,05827	C#	1,05588
D	1,11993	D	1,11740	D	1,11993	D	1,11993
Eb/D#	1,18786	Eb/D#	1,18519	Eb/D#	1,19055	Eb/D#	1,18786
E	1,25566	E	1,21859	E	1,25708	E	1,25708
F	1,33333	F	1,33333	F	1,33635	F	1,33333
F#	1,41102	F#	1,40466	F#	1,41262	F#	1,40943
G	1,49662	G	1,49493	G	1,49662	G	1,49662
G#	1,58561	G#	1,58025	G#	1,58740	G#	1,58382
A	1,67610	A	1,67044	A	1,67800	A	1,67610
Bb	1,77979	Bb	1,77778	Bb	1,78381	Bb	1,77979
B	1,88349	B	1,87289	B	1,88562	B	1,88136
C	2,00000	C	2,00000	C	2,00000	C	2,00000

Kelletat		Kellner		Barnes		Billeter I			
C	1,00000	C	1,00000	C	1,00000	C	1,00000		
C#	1,05469	C#	1,05350	C#	1,05588	C#	1,05469		
D	1,11740	D	1,11892	D	1,11993	D	1,12267		
Eb/D#	1,18652	Eb/D#	1,18519	Eb/D#	1,18786	Eb/D#	1,18652		
E	1,25000	E	1,25198	E	1,25424	E	1,25259		
F	1,33484	F	1,33333	F	1,33635	F	1,33484		
F#	1,40625	F#	1,40466	F#	1,40784	F#	1,40625		
G	1,49493	G	1,49594	G	1,49662	G	1,49845		
G#	1,58203	G#	1,58025	G#	1,58382	G#	1,58203		
A	1,67044	A	1,67384	A	1,67610	A	1,67705		
Bb	1,77979	Bb	1,77778	Bb	1,78180	Bb	1,77979		
B	1,87500	B	1,87797	B	1,88136	B	1,87694		
C	2,00000	C	2,00000	C	2,00000	C	2,00000		
Lindley I		**Lindley II – Average Neidhardt**		**Sparschuh**		**Jira I**			
C	1,00000	C	1,00000	C	1,00000	C	1,00000		
C#	1,05561	C#	1,05648	C#	1,05600	C#	1,05231		
D	1,12011	D	1,11993	D	1,12000	D	1,11993		
Eb/D#	1,18677	Eb/D#	1,18719	Eb/D#	1,18800	Eb/D#	1,18519		
E	1,25464	E	1,25495	E	1,25600	E	1,25424		
F	1,33520	F	1,33409	F	1,33400	F	1,33333		
F#	1,40846	F#	1,40943	F#	1,40800	F#	1,40784		
G	1,49673	G	1,49662	G	1,49600	G	1,49662		
G#	1,58230	G#	1,58382	G#	1,58400	G#	1,57846		
A	1,67650	A	1,67610	A	1,68000	A	1,67610		
Bb	1,78021	Bb	1,77979	Bb	1,78000	Bb	1,77778		
B	1,87926	B	1,88136	B	1,88000	B	1,88136		
C	2,00000	C	2,00000	C	2,00000	C	2,00000		
Jira II		**Zapf**		**Francis I**		**Lehman I**		**Lehman II**	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05588	C#	1,05959	C#	1,05906	C#	1,05906	C#	1,05827
D	1,11993	D	1,12000	D	1,12500	D	1,12035	D	1,11993
Eb/D#	1,18652	Eb/D#	1,18954	Eb/D#	1,19145	Eb/D#	1,18887	Eb/D#	1,18786
E	1,25424	E	1,25700	E	1,26040	E	1,25519	E	1,25424
F	1,33333	F	1,33574	F	1,34038	F	1,33610	F	1,33635
F#	1,40943	F#	1,41412	F#	1,41209	F#	1,41209	F#	1,41102
G	1,49662	G	1,49600	G	1,50000	G	1,49690	G	1,49662
G#	1,58203	G#	1,58739	G#	1,59519	G#	1,58695	G#	1,58561
A	1,67610	A	1,67600	A	1,68750	A	1,67705	A	1,67610
Bb	1,77778	Bb	1,78231	Bb	1,78717	Bb	1,78146	Bb	1,77979
B	1,88136	B	1,88550	B	1,88278	B	1,88278	B	1,88136
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000

Francis II – Cammerton		Francis III – Cornet-ton		Dent I		Dent II		Dent III	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05654	C#	1,05985	C#	1,05687	C#	1,05687	C#	1,05687
D	1,12149	D	1,12053	D	1,12035	D	1,12035	D	1,12035
Eb/D#	1,18861	Eb/D#	1,19009	Eb/D#	1,18775	Eb/D#	1,18775	Eb/D#	1,18775
E	1,25665	E	1,25612	E	1,25649	E	1,25519	E	1,25649
F	1,33467	F	1,33572	F	1,33622	F	1,33622	F	1,33622
F#	1,40872	F#	1,41313	F#	1,41062	F#	1,41062	F#	1,41062
G	1,49799	G	1,49642	G	1,49690	G	1,49690	G	1,49690
G#	1,58481	G#	1,58798	G#	1,58367	G#	1,58367	G#	1,58367
A	1,67822	A	1,67721	A	1,67705	A	1,67705	A	1,67705
Bb	1,78090	Bb	1,78335	Bb	1,78163	Bb	1,78163	Bb	1,78163
B	1,88097	B	1,88417	B	1,88278	B	1,88278	B	1,88278
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000
Ponsford I		**Ponsford II**		**Maunder I**		**Maunder II**		**Maunder III**	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05946	C#	1,06546	C#	1,05827	C#	1,05827	C#	1,05963
D	1,12120	D	1,12500	D	1,11993	D	1,11993	D	1,12065
Eb/D#	1,18921	Eb/D#	1,19324	Eb/D#	1,18876	Eb/D#	1,18921	Eb/D#	1,18978
E	1,25566	E	1,26277	E	1,25424	E	1,25424	E	1,25586
F	1,33484	F	1,33635	F	1,33635	F	1,33635	F	1,33592
F#	1,41262	F#	1,42062	F#	1,41102	F#	1,41102	F#	1,41285
G	1,49831	G	1,50169	G	1,49662	G	1,49662	G	1,49710
G#	1,58740	G#	1,59459	G#	1,58621	G#	1,58651	G#	1,58791
A	1,67800	A	1,68560	A	1,67610	A	1,67610	A	1,67773
Bb	1,78180	Bb	1,78583	Bb	1,78180	Bb	1,78280	Bb	1,78295
B	1,88349	B	1,89415	B	1,88136	B	1,88136	B	1,88379
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000
Mobbs-Mackenzie		**Lucktenberg**		**Jencka**		**Lehman III**		**Lindley-Ortgies I**	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05576	C#	1,05887	C#	1,05827	C#	1,05588	C#	1,05707
D	1,12343	D	1,12120	D	1,11993	D	1,11993	D	1,12120
Eb/D#	1,18773	Eb/D#	1,18921	Eb/D#	1,18876	Eb/D#	1,18786	Eb/D#	1,18786
E	1,25907	E	1,25708	E	1,25424	E	1,25424	E	1,25708
F	1,33429	F	1,33559	F	1,33635	F	1,33333	F	1,33559
F#	1,40768	F#	1,41182	F#	1,41102	F#	1,41102	F#	1,41102
G	1,49896	G	1,49746	G	1,49662	G	1,49662	G	1,49746
G#	1,58364	G#	1,58561	G#	1,58621	G#	1,58025	G#	1,58471
A	1,68398	A	1,67895	A	1,67610	A	1,67610	A	1,67895
Bb	1,78032	Bb	1,78079	Bb	1,78180	Bb	1,78180	Bb	1,78079
B	1,88274	B	1,88243	B	1,88136	B	1,88136	B	1,88349
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000

Lindley-Ortgies II		O'Donnell		Spányi		Interbartolo-Venturino I		Interbartolo-Venturino II	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05588	C#	1,05707	C#	1,05447	C#	1,05469	C#	1,05350
D	1,11993	D	1,11993	D	1,12500	D	1,11803	D	1,11740
Eb/D#	1,18652	Eb/D#	1,18652	Eb/D#	1,18583	Eb/D#	1,18809	Eb/D#	1,18786
E	1,25424	E	1,25708	E	1,25231	E	1,25000	E	1,24859
F	1,33183	F	1,33333	F	1,33356	F	1,33784	F	1,33786
F#	1,40784	F#	1,41102	F#	1,40885	F#	1,40625	F#	1,40466
G	1,49662	G	1,49662	G	1,50000	G	1,49535	G	1,49493
G#	1,58382	G#	1,58382	G#	1,58141	G#	1,58307	G#	1,58203
A	1,67610	A	1,67610	A	1,67860	A	1,67185	A	1,67044
Bb	1,77778	Bb	1,77778	Bb	1,77841	Bb	1,78331	Bb	1,78381
B	1,87712	B	1,88136	B	1,87846	B	1,87500	B	1,87289
C	2,00000	C	2,00000	C	**1,99996**	**C**	2,00000	C	2,00000
Interbartolo-Venturino III		**Jobin**		**Di Veroli I**		**Di Veroli II**		**Glueck**	
C	1,00000	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,05350	C#	1,05142	C#	1,05707	C#	1,05707	C#	1,05887
D	1,11803	D	1,11803	D	1,11993	D	1,11993	D	1,12120
Eb/D#	1,18764	Eb/D#	1,18362	Eb/D#	1,18921	Eb/D#	1,18921	Eb/D#	1,18854
E	1,25000	E	1,25000	E	1,25566	E	1,25566	E	1,25708
F	1,33784	F	1,33333	F	1,33635	F	1,33635	F	1,33559
F#	1,40625	F#	1,40189	F#	1,40943	F#	1,40943	F#	1,41262
G	1,49535	G	1,49535	G	1,49662	G	1,49662	G	1,49746
G#	1,58188	G#	1,57713	G#	1,58561	G#	1,58740	G#	1,58651
A	1,67185	A	1,67185	A	1,67800	A	1,67800	A	1,67895
Bb	1,78331	Bb	1,77661	Bb	1,78381	Bb	1,78381	Bb	1,78079
B	1,87500	B	1,86919	B	1,88136	B	1,88136	B	1,88455
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000
Briggs		**Billeter IIa**		**Billeter IIb**		**Billeter IIIa**		**Billeter IIIb**	
C	1,00682	C	1,00000	C	1,00000	C	1,00000	C	1,00000
C#	1,06379	C#	1,05350	C#	1,05469	C#	1,05797	C#	1,05588
D	1,12254	D	1,11930	D	1,11930	D	1,11993	D	1,12025
Eb/D#	1,19553	Eb/D#	1,18518	Eb/D#	1,18652	Eb/D#	1,18820	Eb/D#	1,18652
E	1,26137	E	1,25283	E	1,25283	E	1,25389	E	1,25460
F	1,34374	F	1,33333	F	1,33333	F	1,33597	F	1,33333
F#	1,41905	F#	1,40625	F#	1,40625	F#	1,41062	F#	1,40784
G	1,49803	G	1,49619	G	1,49662	G	1,49662	G	1,49704
G#	1,59470	G#	1,58025	G#	1,58203	G#	1,58516	G#	1,58382
A	1,68183	A	1,67469	A	1,67516	A	1,67610	A	1,67658
Bb	1,79231	Bb	1,77778	Bb	1,77778	Bb	1,78129	Bb	1,77878
B	1,89206	B	1,87924	B	1,87924	B	1,88083	B	1,87712
C	2,00000	C	2,00000	C	2,00000	C	2,00000	C	2,00000

Table 3: Results obtained

BWV	846	848	850	852	854	856	858	860	862	864	866	868	AVERAGE	STANDARD DEVIATION
Historical temperaments														
Kirnberger II	-27,51	-0,83	-2,21	-3,72	4,29	-5,90	1,94	-26,88	-3,53	-6,33	-7,96	-0,04	-6,56	10,25
Kirnberger III	-9,32	-0,42	3,91	-3,51	3,01	-10,45	1,39	-3,17	-3,08	0,85	-9,94	-0,38	-2,59	4,99
Werckmeister III	-3,09	-0,27	4,11	-3,37	-1,38	-11,44	1,81	2,75	-3,22	-0,73	-10,21	2,40	-1,89	4,86
Tartini - Vallotti	-4,33	-0,29	-0.23	-4,74	4,11	-5,81	0,09	-4,16	-4,05	-0,33	-7,23	-0,26	-2,27	3,25
Neidhardt 1732 - 5th circle XI	-2,72	6,15	9,74	5,30	5,68	-1,42	5,89	3,47	8,45	6,80	1,30	5,26	4,49	3,74
Neidhardt 1732 - 3rd circle IV	-0,53	4,21	2,76	2,57	1,03	-3,21	4,65	0,96	3,87	0,99	-1,01	1,24	1,46	2,32
Neidhardt 1732 - 3rd circle II - Small city = Neidhardt 1724 I - Village	-4,41	-1,73	1,15	0,14	3,37	-5,06	-0,28	-2,07	0,23	-0,16	-5,34	1,13	-1,09	2,71
Sorge 1744 III	-0,68	-2,19	1,67	-0,74	0,52	-4,29	2,24	0,76	-2,62	-0,68	-3,32	2,01	-0,61	2,15
Bach temperaments														
Kelletat	-8,20	-0,76	5,00	-5,24	2,77	-12,29	-0,52	-1,74	-4,09	1,72	-11,35	-1,76	-3,04	5,42
Kellner	-6,55	-0,31	0,92	-2,76	1,80	-8,07	1,76	-2,14	-3,01	-1,76	-8,23	2,26	-2,17	3,76
Barnes	-2,79	-0,42	1,53	-4,49	1,67	-5,73	1,42	-0,31	-4,05	-0,45	-6,64	1,54	-1,56	3,05
Billeter I	-13,56	-0,56	-0,51	-3,16	4,53	-3,97	1,16	-13,51	-3,34	-3,13	-5,98	1,07	-3,41	5,50
Lindley I	-3,39	2,44	0,81	-2,92	2,72	-4,98	1,93	-2,19	-1,99	-0,07	-6,09	0,99	-1,06	2,95
Lindley II – Average Neidhardt	-2,41	0,54	2,87	-1,27	1,81	-4,45	1,04	0,32	-1,04	0,83	-4,57	0,71	-0,47	2,35
Sparschuh	-1,37	-1,90	-7,23	-1,47	3,25	-3,50	0,99	-2,78	-2,75	-3,39	-3,60	1,92	-1,82	2,82
Jira I	-3,73	4,58	0,96	0,01	-0,69	-6,28	9,54	-0,35	2,72	-3,18	-6,33	6,62	0,32	4,93
Jira II	-3,50	2,40	3,10	-0,41	1,01	-5,00	1,16	0,07	-0,66	0,66	-5,15	0,38	-0,50	2,69
Zapf	2,02	0,86	5,51	-2,33	-1,00	-4,12	-1,15	3,10	-1,24	2,68	-3,72	-2,44	-0,15	2,98
Briggs	3,76	0,49	1,01	-10,52	-0,75	-15,22	-0,89	2,36	-5,55	2,03	-15,85	-2,30	-3,45	6,83
Francis I	-6,63	-0,56	-12,31	-4,85	13,66	0,28	12,77	-13,57	4,09	0,18	-3,34	11,95	0,14	9,16
Lehman I	-1,60	1,00	3,74	-1,82	2,17	-3,84	-2,06	0,47	-0,30	2,38	-4,49	-2,35	-0,56	2,57
Lehman II	-2,33	4,49	3,78	2,07	2,45	-2,67	0,12	-0,01	3,45	2,39	-0,37	-1,94	0,95	2,49
Francis II - Cammerton	-3,98	-2,14	-0,08	-1,15	2,80	-2,23	1,07	-3,56	-2,98	-1,46	-2,58	1,83	-1,21	2,17
Francis III - Cornet-ton	-0,05	-0,22	4,38	-2,75	1,43	-2,46	-1,81	0,95	-2,40	2,56	-2,65	-2,75	-0,48	2,40
Dent I	-0,18	3,10	2,21	-3,73	0,58	-4,54	1,92	0,98	-2,44	0,69	-5,63	0,68	-0,53	2,85
Dent II	-1,58	3,08	2,84	-3,82	0,95	-4,50	1,78	0,59	-2,52	1,05	-5,70	0,16	-0,64	2,93

Dent III	-0,18	3,10	2,21	-3,73	0,58	-4,54	1,92	0,98	-2,44	0,69	-5,63	0,68	-0,53	2,85
Ponsford I	-4,01	-0,39	4,25	-0,43	2,32	-1,76	-2,14	-0,25	-0,55	2,21	-1,63	-2,56	-0,41	2,33
Ponsford II	3,81	-5,31	2,35	2,23	-0,61	5,75	-8,64	1,76	-0,42	2,05	7,06	-6,26	0,31	4,83
Maunder I	-2,54	0,77	3,84	-2,99	2,52	-5,06	-1,30	0,01	-1,53	2,22	-5,71	-2,01	-0,98	2,97
Maunder II	-2,64	-0,15	3,88	-3,66	2,56	-4,74	-0,89	0,02	-2,75	2,12	-5,18	-2,00	-1,12	2,91
Maunder III	-1,19	-0,28	3,70	-2,21	2,01	-2,49	-1,79	0,74	-1,65	2,34	-2,76	-2,47	-0,50	2,19
Mobbs-Mackenzie	-5,56	-1,40	-11,50	-1,06	0,77	-0,84	2,56	-9,43	-2,67	-9,24	-1,81	3,72	-3,04	4,88
Lucktenberg	-1,23	2,69	0,90	-1,01	2,53	-2,17	-1,68	-1,39	-1,12	1,22	-3,02	-1,57	-0,49	1,86
Jencka	-2,54	0,77	3,84	-2,99	2,52	-5,06	-1,30	0,01	-1,53	2,22	-5,71	-2,01	-0,98	2,97
Lehman III	-3,55	12,86	4,17	1,74	2,92	-3,65	11,06	0,38	7,16	1,92	-2,80	5,40	3,13	5,37
Lindley-Ortgies I	-0,96	1,08	0,68	-1,72	1,34	-2,78	1,72	-0,52	-0,95	-0,15	-3,81	1,31	-0,40	1,75
Lindley-Ortgies II	-3,43	-1,70	0,33	3,30	4,45	-0,83	-0,36	-4,12	2,49	-0,33	1,42	-0,07	0,10	2,55
O'Donnell	-0,38	1,61	2,62	0,61	0,59	-4,98	-0,61	0,86	1,64	0,99	-4,80	-0,52	-0,20	2,39
Spányi	-20,35	-0,16	0,96	-2,53	3,35	-3,22	2,75	-19,49	-2,36	-3,91	-5,68	0,31	-4,19	7,82
Interbartolo-Venturino I	-8,68	7,97	3,22	-3,02	5,97	-11,31	7,09	-3,40	1,98	2,38	-9,21	3,57	-0,29	6,67
Interbartolo-Venturino II	-7,21	12,71	3,07	-1,81	7,97	-10,15	11,16	-3,92	5,58	2,86	-10,13	6,89	1,42	7,97
Interbartolo-Venturino III	-8,68	12,48	3,22	-0,70	6,75	-11,44	12,51	-3,25	5,94	2,73	-8,41	7,51	1,56	8,13
Billeter IIa	-5,31	-0,03	1,60	-2,46	1,03	-7,15	3,05	-1,21	-2,66	-1,48	-7,49	2,50	-1,63	3,56
Billeter IIb	-5,91	-1,26	0,97	-0,89	2,21	-6,54	1,08	-1,19	-2,16	-1,11	-6,12	1,93	-1,58	3,10
Billeter IIIa	-3,20	2,28	3,57	-2,85	2,63	-5,04	-1,14	-0,55	-1,27	2,14	-5,81	-2,14	-0,95	3,07
Billeter IIIb	-5,74	-1,81	-0,42	-0,52	4,47	-4,36	-0,44	-4,86	-0,34	-0,72	-4,95	0,16	-1,63	2,91
Jobin	-12,58	5,48	-3,56	2,40	5,99	-9,50	6,17	-11,16	4,46	-3,06	-6,71	3,95	-1,51	7,12
Di Veroli I	-1,01	-1,61	-1,09	-4,98	2,82	-4,01	1,05	0,03	-5,11	-0,29	-4,67	0,66	-1,52	2,62
Di Veroli II	-1,09	4,52	-0,88	-3,93	5,79	-3,96	5,06	0,08	-1,35	1,69	-4,54	4,33	0,48	3,74
Glueck	-0,55	1,39	2,20	-0,68	1,54	-2,27	-0,96	0,37	0,51	1,53	-3,05	-0,83	-0,07	1,61

Table 4: Key signatures in Bach's clavier and organ works

KEY	CLAVIER	ORGAN	BOTH
No sharps / flats	105	111	216
1-sharp	97	84	181
2-sharps	60	32	92
3-sharps	30	17	47
4-sharps	23	6	29
5-sharps / 7-flats	9	0	9
6-sharps / 6-flats	9	0	9
7-sharps / 5-flats	10	0	10
4-flats	10	0	10
3-flats	45	38	83
2-flats	61	41	102
1-flat	77	59	136

Table 5: Correlations

TEMPERAMENT	CLAVIER	ORGAN	BOTH
Historical temperaments			
Kirnberger II	-0,76	-0,73	-0,75
Kirnberger III	-0,97	-0,96	-0,97
Werckmeister III	-0,90	-0,85	-0,88
Tartini - Vallotti	-0,96	-0,89	-0,93
Neidhardt 1732 - 5th circle XI	-0,53	-0,57	-0,56
Neidhardt 1732 - 3rd circle IV	-0,89	-0,83	-0,87
Neidhardt 1732 - 3rd circle II - Small city = Neidhardt 1724 I - Village	-0,96	-0,93	-0,95
Sorge 1744 III	-0,95	-0,87	-0,91

Bach temperaments			
Kelletat	-0,96	-0,94	-0,96
Kellner	-0,92	-0,87	-0,90
Barnes	-0,92	-0,86	-0,90
Billeter I	-0,87	-0,81	-0,85
Lindley I	-0,95	-0,89	-0,93
Lindley II - Average Neidhardt	-0,98	-0,95	-0,97
Sparschuh	-0,85	-0,80	-0,83
Jira I	-0,70	-0,65	-0,68
Jira II	-0,92	-0,91	-0,92
Zapf	-0,51	-0,54	-0,53
Briggs	-0.57	-0.60	-0.59
Francis I	-0,44	-0,36	-0,40
Lehman I	-0,83	-0,84	-0,84
Lehman II	-0,88	-0,89	-0,89
Francis II - Cammerton	-0,91	-0,82	-0,87
Francis III - Cornet-ton	-0,67	-0,69	-0,68
Dent I	-0,89	-0,83	-0,87
Dent II	-0,92	-0,88	-0,91
Dent III	-0,89	-0,83	-0,87
Ponsford I	-0,81	-0,83	-0,83
Ponsford II	0,55	0,48	0,52
Maunder I	-0,88	-0,88	-0,89
Maunder II	-0,87	-0,86	-0,87
Maunder III	-0,75	-0,77	-0,76
Mobbs-Mackenzie	-0,39	-0,29	-0,34
Lucktenberg	-0,94	-0,91	-0,93
Jencka	-0,88	-0,88	-0,89
Lehman III	-0,91	-0,89	-0,91
Lindley-Ortgies I	-0,91	-0,85	-0,88
Lindley-Ortgies II	-0,93	-0,91	-0,92
O'Donnell	-0,85	-0,83	-0,85
Spányi	-0,71	-0,68	-0,70
Interbartolo-Venturino I	-0,93	-0,89	-0,92
Interbartolo-Venturino II	-0,94	-0,89	-0,92
Interbartolo-Venturino III	-0,93	-0,89	-0,92
Billeter IIa	-0,89	-0,84	-0,87
Billeter IIb	-0,95	-0,91	-0,94
Billeter IIIa	-0,93	-0,92	-0,93
Billeter IIIb	-0,95	-0,90	-0,93
Jobin	-0,94	-0,88	-0,91
Di Veroli I	-0.89	-0.83	-0.87
Di Veroli II	-0,85	-0,80	-0,83
Glueck	-0,88	-0,89	-0,89